Reliability of connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): F lattice systems

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Abstract. A three dimensional consecutive \(k\)-out-of-\(n\): F system is a cuboid grid of edges \(n_1,n_2,n_3\) containing \(n_1 \times n_2 \times n_3\) components and fails if and only if there is at least a cuboid of edges \(r_1,r_2,r_3\), where \(r_1 \leq r_1, r_2 \leq n_2, r_3 \leq n_3\), containing \(r_1,r_2,r_3\) of failed components. Few researchers set numerous algorithms to compute the reliability of special cases of such systems.

A connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): F lattice system, consists of \(4n\) components, arranged in a cuboid of \(n\) layers, each layer consists of \(2 \times 2\) components. This system fails if and only if it contains at least 2 connected failed components.

In this paper, the reliability of the connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): F lattice system is obtained utilizing a general formula that is driven using a Markov chain technique and a recursive algorithm.

Keywords: Three dimensional consecutive \((r_1,r_2,r_3)\)-out-of-\((n_1,n_2,n_3)\): F system; Connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): F lattice system; Markov chain; Recursive algorithm.

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1 Introduction

The consecutive \(k\)-out-of-\(n\): F system and its generalizations have attracted considerable attention in a huge number of papers ([1], [4], [5] and the references therein). A consecutive \(k\)-out-of-\(n\): F system consists of a sequence of \(n\) ordered components

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which either fail or operate. The system fails whenever \( k \) consecutive components are failed, \( 1 < k < n \). Until now, a large number of closed formulas, recursive and direct algorithms, limit formulas and bounds for the reliability of linear and circular consecutive \( k \)-out-of-\( n \): \( F \) system are established by use of methods from combinatorics, probability theory, switching-algebra, graph theory and so on.

The two dimensional generalization of the consecutive \( k \)-out-of-\( n \): \( F \) system is considered by several authors. Such system consists of \( n^2 \) components in a square grid of side \( n \) and it fails if and only if there is at least one square of side \( k \) \( (k < n) \) that contains all failed components. Closed reliability formulas are only known for some simple cases ([6]). A step forward generalization for the two dimensional consecutive \( k \)-out-of-\( n \): \( F \) system is the three dimensional consecutive \( (r_1, r_2, r_3) \)-out-of-\( (n_1, n_2, n_3) \): \( F \) system. A three dimensional consecutive \( (r_1, r_2, r_3) \)-out-of-\( (n_1, n_2, n_3) \): \( F \) system is a cuboid grid of edges \( n_1, n_2, n_3 \) containing \( n_1.n_2.n_3 \) components and fails if and only if there is at least a cuboid of edges \( r_1, r_2, r_3, r_1 \leq n_1, r_2 \leq n_2, r_3 \leq n_3 \) containing \( r_1.r_2.r_3 \) failed components. The computation of reliability in the three-dimensional system is more complicated than the other in the lower dimensional systems, but sometimes it is more tractable. Up to now some researchers studied multi-dimensional consecutive \( k \)-out-of-\( n \): \( F \) systems, and showed promising applications of such multi-dimensional models, e.g., diagnosis of a disease diagnosed by reading an \( X \)-ray. As another example, the three-dimensional system can be applied for the mathematical model of a three-dimensional flash memory cell failure model ([1]). Few researchers studied the three-dimensional consecutive \( k \)-out-of-\( n \): \( F \) system. Most of them studied some special cases of the three-dimensional systems. Akiba et al. [2] studied the three-dimensional adjacent triangle: \( F \) triangular lattice system, this system can be applied as the mathematical model of a scatter water area of a water sprinkler system etc., also Akiba et al. [1] studied the lower and upper bounds of the \( k \)-within- consecutive-\( (r_1, r_2, r_3) \)-out-of-\( (n_1, n_2, n_3) \): \( F \) system.

A number of researchers set numerous algorithms to compute the reliability for some special cases of such systems. An important special case from the three dimensional consecutive \( (r_1, r_2, r_3) \)-out-of-\( (n_1, n_2, n_3) \): \( F \) system is the connected \( (1,1,2) \) or \( (1,2,1) \) or \( (2,1,1) \)-out-of-\( (n,2,2) \): \( F \) lattice system. A connected \( (1,1,2) \) or \( (1,2,1) \) or \( (2,1,1) \)-out-of-\( (n,2,2) \): \( F \) lattice system, consists of \( 4n \) components, arranged in a cuboid of \( n \) layers, each layer consists \( 2 \times 2 \) of components and fails if and only if there is at least 2 connected failed components in the \( (n,2,2) \) cuboid. Figures 1.a and 1.b show some functioning and failed states of this system.
In this article, the reliability of the connected $(1,1,2)$ or $(1,2,1)$ or $(2,1,1)$-out-of-$(n,2,2)$: $F$ lattice system is obtained utilizing a general formula that is driven using a Markov chain technique and a recursive algorithm.

**Notations**

- $R(n)$: Reliability of the connected $(1,1,2)$ or $(1,2,1)$ or $(2,1,1)$-out-of-$(n,2,2)$: $F$ lattice system
- $(0,1)$: Represents the functioning and failed component respectively
- $[x]$: The greatest integer lower bound of $x$
- $p, q$: Components reliability and unreliability respectively, where $p + q = 1$
- $P = [P_{ij}]$: Transition probability matrix
- $\chi_u$: Indicator function

The following assumptions are assumed to be satisfied by the connected $(1,1,2)$ or $(1,2,1)$ or $(2,1,1)$-out-of-$(n,2,2)$: $F$ lattice system:

1. The state of the component as well as the state of the system is either ”functioning” or ”failed”.
2. The components are independent and identically distributed.
3. The system fails if there are at least 2 connected failed components.

## 2 Evaluating the reliability of the system

### 2.1 A general formula for the system reliability

Consider a system in the functioning state that consists of $n$ layers each of $2 \times 2$ components, such that each layer has 2 failed components at maximum, otherwise the system fails. Represent this system by an $n$-digits number, such that each digit represents the number of failed components in the corresponding layer. Consequently each digit is either 0,1, or 2, for example, the three digits numbers 120, and 111 represent, respectively, the state of the system as shown in figure 2.a, while, the numbers 110, and 121 represent the state of the system as shown in figure 2.b.
Theorem 2.1. The reliability of the connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): \(F\) system is:

\[
R(n) = \sum_{j=1}^{n} L(n,j)p^{4n-j}q^j
\]

Where \(L(n,j)\) denotes the number of non-zero configurations of components having \(j\) total of failed components without two connected failures. \(L(n,j)\) for \(n \leq 6\) are given in table 1 below.

<table>
<thead>
<tr>
<th>(n/j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>16</td>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>12</td>
<td>46</td>
<td>92</td>
<td>184</td>
<td>92</td>
<td>46</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>16</td>
<td>92</td>
<td>272</td>
<td>976</td>
<td>488</td>
<td>272</td>
<td>112</td>
<td>50</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>20</td>
<td>154</td>
<td>588</td>
<td>1870</td>
<td>1184</td>
<td>588</td>
<td>292</td>
<td>128</td>
<td>60</td>
<td>30</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>24</td>
<td>232</td>
<td>1176</td>
<td>5440</td>
<td>2336</td>
<td>1176</td>
<td>588</td>
<td>294</td>
<td>140</td>
<td>68</td>
<td>34</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 1: \(L(n,j)\) for consecutive \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-(\(n,2,2)\): \(F\) system.

In example 1 below, the reliability of the connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-(\(2,2,2)\): \(F\) lattice system is calculated.

Example 2.1. The reliability of the connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-(\(2,2,2)\): \(F\) lattice system.

<table>
<thead>
<tr>
<th>Number of Configurations</th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>02</th>
<th>20</th>
<th>11</th>
<th>12</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reliability</td>
<td>(p^8)</td>
<td>(4p^7q)</td>
<td>(4p^6q^2)</td>
<td>(2p^5q^2)</td>
<td>(2p^4q^3)</td>
<td>(12p^3q^4)</td>
<td>(4p^2q^5)</td>
<td>(4p^3q^4)</td>
<td>(2p^4q^4)</td>
</tr>
</tbody>
</table>

\[
R(n) = p^8 + 8p^7q + 16p^6q^2 + 8p^5q^3 + 2p^4q^4
\]

2.2 Computing the reliability of the system using a Markov chain technique

Consider the system in the functioning state with \(n\) layers each of \(2 \times 2\) components. Let \(X_k\) be a random variable that counts the number of failed components at the \(k^{th}\) layer \((X_k \in \{0,1,2\})\), where the failed components in the \(k + 1^{th}\) layer must not be connected to those in the \(k^{th}\) layer.
Note that the number of failed components in the $k + 1^{th}$ layer depends only on the failed components on the $k^{th}$ layer. Therefore the random variable $X_{k+1}$ depends only on $X_k$ but not on $X_{k-1}, X_{k-2}, ..., X_1$, hence the sequence $\{X_k\}, k = 1, 2, ..., n$ forms a Markov chain.

**Lemma 2.2.** If the connected $(1, 1, 2)$ or $(1, 2, 1)$ or $(2, 1, 1)$-out-of-$(n, 2, 2)$: $F$ lattice system is in the functioning state, and the variables $\{X_k\}, k = 1, 2, ..., n$ count the number of failed components in the $k^{th}$ layer, then the reliability of the system can be expressed as:

$$R(n) = \sum_{j=0}^{2} P_{0j}^n = \sum_{j=0}^{2} P^n(X_{k+1} = j | X_k = 0)$$

Where $P_{ij}^n$ is the $n$-step transition probability from state $i$ to state $j$ and the one step transition probability matrix of the corresponding Markov chain is given by:

$$P = \begin{pmatrix} p^4 & 4pq^3 & 2pq^2 \\ p^4 & 3pq^3 & pq^2 \\ p^4 & 2pq^3 & q^2 \\ \end{pmatrix}$$

The second equality in the above expression for $R(n)$ is based on the well-known result in the theory of discrete homogeneous Markov chains that the $n$-step transition probability matrix equals the $n^{th}$ power of the corresponding one step transition probability matrix ([7]).

**Example 2.2.** The reliability of the connected $(1, 1, 2)$ or $(1, 2, 1)$ or $(2, 1, 1)$-out-of-$(n, 2, 2)$: $F$ lattice system can be expressed as:

$$R(n) = \sum_{j=0}^{2} P_{0j}^n = p^8 + 8p^7q + 16p^6q^2 + 8p^5q^3 + 2p^4q^4$$

where the

$$P^2 = \begin{pmatrix} p^8 + 4p^7q + 2p^6q^2 & 4p^7q + 12p^6q^2 + 4p^5q^3 & 2p^6q^2 + 4pq^3 + 2p^4q^4 \\ p^8 + 3p^7q + p^6q^2 & 4p^7q + 9p^6q^2 + 2p^5q^3 & 2p^6q^2 + 3p^5q^3 + p^4q^4 \\ p^8 + 2p^7q + p^6q^2 & 4p^7q + 6p^6q^2 + 2p^5q^3 & 2p^6q^2 + 2p^5q^3 + p^4q^4 \\ \end{pmatrix}$$

In the following section, we transform the shape of the system to an applicable form that makes possible to apply the recursive algorithm concerning the computation of the reliability of the connected $(1, 2)$ or $(2, 1)$-out-of-$(n, m)$: $F$ circular system given in [8], to compute the reliability of the system.

### 2.3 A recursive algorithm for computing the reliability of the system

Consider the system in the functioning state. Make a vertical cut of the connected $(1, 1, 2)$ or $(1, 2, 1)$ or $(2, 1, 1)$-out-of-$(n, 2, 2)$: $F$ lattice system between the 4th and the 1st knots, rotate the graph 90 degrees with clockwise direction as in figure 2 below,
the resulting shape is equivalent to the two dimensional connected \((1,2)\) or \((2,1)\)-out-of-\((n,4)\): \(F\) lattice circular system, which implies that the reliability of the \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): \(F\) equals the reliability of \((1,2)\) or \((2,1)\)-out-of-\((n,4)\): \(F\) circular system.

Now, let \((i,j)\) denote the location of the component in the connected \((1,2)\) or \((2,1)\)-out-of-\((n,4)\): \(F\) circular system, where \(i = 1,2,3,4\). Let \(Z_{ij}\) be the location variable, which determines the status of the component in the column \(j\) and row \(i\), such that:

\[
Z_{ij} = \begin{cases} 
1 & \text{Component fail} \\
0 & \text{Component function}
\end{cases}
\]

Where \(i = 1,2,3,4,5\), \(j = 1,2,...,n\) and \(Z_{1,j} = Z_{5,j}\).

Let \(A_{i,j}\) be the event of the resulted failure from the component \((i,j)\) then

\[
A_{i,j} = \{Z_{i,j} = 1\} \cap \{Z_{i,j-1} = 1\} \cup \{Z_{i-1,j} = 1\}
\]

Where \(A_{5,j} = A_{1,j}\), hence the reliability for the component \((i,j)\) is represented by \(A_{i,j}^c\), and this implies that the reliability of the system can be written as:

\[
R(n) = P\left\{\bigcap_{u=1}^{5} \bigcap_{v=1}^{n} A_{u,v}^c\right\}.
\]

For any column \(j\), let \(\Theta\) be the set of 4-dimensional binary vectors with 0 indicating functioning and 1 for failed state, i.e. \(\Theta = \{y = (y_1, y_2, y_3, y_4) : y_i y_{i-1} = 0 \land y_1 y_4 = 0 : i = 1,2,3,4\}\)

\[
\Theta = \{(0101), (1010), (1000), (0100), (0010), (0001), (0000)\}
\]

As in [8] for any binary vector \(y \in \Theta\) define the set

\[
\Omega(y) = \left\{(x_1, x_2, ..., x_n) \in \Theta : \bigcap_{u=1}^{5} A_{u,j} \bigcap \bigcap_{u=1}^{5} Z_{u,j} = \chi_u \bigcap \{Z_{u,j-1} = \chi_u\} \neq \phi \right\}
\]

\[
\Omega(0000) = \{(0101), (1010), (1000), (0100), (0010), (0001), (0000)\}
\]

\[
\Omega(1000) = \{(0101), (0100), (0010), (0001), (0000)\}
\]
\(\Omega(0100) = \{(1010), (1000), (0010), (0001), (0000)\}\)
\(\Omega(0010) = \{(0101), (1000), (0100), (0001), (0000)\}\)
\(\Omega(0001) = \{(1010), (1000), (0100), (0010), (0000)\}\)
\(\Omega(1010) = \{(0101), (0100), (0010), (0001), (0000)\}\)
\(\Omega(0101) = \{(1010), (1000), (0010), (0001), (0000)\}\)

And \(R(j)\) to be the reliability of the connected \((1, 2)\) or \((2, 1)\)-out-of-\((j, 4)\): \(F\) lattice system, for \(j = 1, 2, ..., n\) and \(R(j, y)\) to be the reliability of the connected \((1, 2)\) or \((2, 1)\)-out-of-\((j, 4)\): \(F\) lattice system, for \(j = 1, 2, ..., n\) with given components state \(y\) on column \(j\), and \(R(j) = \sum_{y \in \Theta} R(j, y)\), where

\[
R(j, y) = \left\{ \begin{array}{ll}
F_j(y) \sum_{x \in \Omega(y)} & j \geq 2 \\
F_j(y) & j = 1
\end{array} \right. ,
\]

\[
F_j(y) = P\left\{ \bigcap_{u=1}^{5} Z_{u,j} = \chi_u, j = 1, 2, ..., n \right\}
\]

The following is an illustrative example.

**Example 2.3.** The reliability of the connected \((1,1,2)\) or \((1,2,1)\) or \((2,1,1)\)-out-of-\((n,2,2)\): \(F\) lattice system equals the reliability of \((1,2)\) or \((2,1)\)-out-of-\((n,2)\): \(F\) circular system. We have

\[
R(1, (0000)) = F_1(0000) = p^4
\]
\[
R(1, (1000)) = R(1, (0100)) = R(1, (0010)) = R(1, (0001)) = F_1(0000) = p^3 q
\]
\[
R(1, (1010)) = R(1, (0101)) = F_1(0101) = p^2 q^2
\]
\[
R(1) = p^4 + 4p^3 q + 2p^2 q^2 \implies
\]
\[
R(2, (0000)) = F_2(0000) \sum_{x \in \Omega(0000)} R(1, x) = p^4 \left[ p^4 + 4p^3 q + 2p^2 q^2 \right]
\]
\[
R(2, (1000)) = R(2, (0100)) = R(2, (0010)) = R(2, (0001)) = F_2(1000) \sum_{x \in \Omega(1000)} R(1, x) = p^3 q \left[ p^4 + 3p^3 q + p^2 q^2 \right]
\]
\[
R(2, (1010)) = R(2, (0101)) = F_2(1010) \sum_{x \in \Omega(1010)} R(1, x) = p^2 q^2 \left[ p^4 + 2p^3 q + p^2 q^2 \right] \implies
\]
\[
R(2) = p^8 + 8p^7 q + 16p^6 q^2 + 8p^5 q^3 + 2p^4 q^4
\]

**Appendix**

**Theorem 2.1**

\[
R(n) = \sum_{j=1}^{n} L(n, j) p^{4n-j} q^j
\]
Proof. Let \( n \) be a positive integer that represents the number of the cuboid layers, where the components are either in an operating or in a failed state. The system fails if and only if there exists at least 2 connected failed components. That is, associated with any fixed \( n \), there is an \( 1 \times 2n \) array \( L \), where \( L = L(n, j); 0 \leq j \leq 2n \). It is clear that

\[
L(n, j) = \begin{cases} 
1 & j = 0 \\
4n & j = 1 \\
2 & j = 2n 
\end{cases}
\]

If \( j = 2n \) then \( L(n, 2) = L(n - 1, 2) + 2L(n - 2, 2) + 4L(n - 1, 1) + 4 \times 3L(n - 3, 0) \) where the 1\(^{st}\) term is the number of all states which end with 0, the 2\(^{nd}\) term is the number of all states which end with 02, the 3\(^{rd}\) term is the number of all states which end with 01, and the last term is the number of all states which end with 011. Similarly,

\[
L(n, 3) = L(n - 1, 3) + 2L(n - 2, 1) + 4L(n - 2, 2) + 4 \times 3L(n - 3, 1) + 4 \times 3^2L(n - 4, 1)
+ 4 \times 2L(n - 3, 0)
\]

\[
= L(n - 1, 3) + 2L(n - 2, 1) + 4 \sum_{i=1}^{2} 3^iL(n - i - 2, j - i - 1) + 4 \times 2L(n - 3, 0)
\]

Where the 1\(^{st}\) term is the number of all states which end with 0, the 2\(^{nd}\) term is the number of all states which end with 02, the 3\(^{rd}\) term is the number of all states which end with 01, 011, or 0111, and the last term is the number of all states which end with 012,021. Similarly, in general

\[
L(n, j) = L(n - 1, j) + 2 \sum_{i=1}^{[j/2]} L(n - i - 1, j - 2i) + 4 \sum_{i=0}^{j-1} 3^iL(n - i - 2, j - i - 1)
+ 4 \sum_{i=1}^{j-2s+1} \left[ 3^i \left( \begin{array}{c} 2 \\
1 \end{array} \right) + 2 \times 3^{i-1} \left( \begin{array}{c} 2 \\
2 \end{array} \right) \right] L(n - i - 3, j - i - 3) + ...
+ 4 \sum_{i=1}^{j-2s+1} \left[ 3^i \left( \begin{array}{c} s \\
1 \end{array} \right) + 2 \times 3^{i-1} \left( \begin{array}{c} s \\
2 \end{array} \right) + ... + 2^{s-1} \times 3^{i-s+1} \left( \begin{array}{c} s \\
3^s \end{array} \right) \right] L(n - i - s - 1, j - i - 2s + 1)
\]

Where the 1\(^{st}\) term is the number of all states which end with 0, the 2\(^{nd}\) term is the number of all states which end with 02, 022, 0222, or 22222, the 3\(^{rd}\) term is the number of all states which end with 01, 011,0111, or 11111, the Penultimate term is the number of all states which end with 012,021, 0112, 0211, or 21111, and the last term is the sum
of all states which end in 0122, 02122, …, or 22221. Therefore

\[ L(n, j) = L(n - 1, j) + 2 \sum_{l=0}^{[j/2]} L(n - l, j - 2l) \]

\[ + 4 \sum_{l=1}^{[j+1/2]} \sum_{i=0}^{[j-2l+1]} \sum_{k=1}^{l} 3^{i-k+1} 2^{k-1} \binom{l}{k} \binom{i}{k-1} L(n - l - 1, j - i - 2l + 1) \]

Where \( L(-1, 0) = 1, L(0, 0) = 1, L(-i, 0) = 0, i > 1, L(n, j) = 0 \) if \( j > 2n \)

\[ \square \]

**Lemma 2.2**

\[ R(n) = \sum_{j=0}^{2} P^n_{ij} = \sum_{j=0}^{2} P^n(X_{k+1} = j | X_{k+1} = 0) \]

*Proof.*

\[ P_{ij} = P(X_{k+1} = j | X_k = i) = \frac{P(X_{k+1} = j, X_k = i)}{P(X_k = i)} \]

\[ P_{00} = \frac{P(X_{k+1} = 0, X_k = 0)}{P(X_k = 0)} = \frac{p^6}{p^5} = p \]

\[ P_{01} = \frac{P(X_{k+1} = 1, X_k = 0)}{P(X_k = 0)} = \frac{4p^7q}{p^4} = 4p^3q \]

\[ P_{02} = \frac{P(X_{k+1} = 2, X_k = 0)}{P(X_k = 0)} = \frac{2p^6q^2}{p^3} = 2p^3q^2 \]

\[ P_{10} = \frac{P(X_{k+1} = 0, X_k = 1)}{P(X_k = 1)} = \frac{4p^7q}{4p^4q} = p \]

\[ P_{11} = \frac{P(X_{k+1} = 1, X_k = 1)}{P(X_k = 1)} = \frac{12p^6q^2}{4p^4q} = 3p^3q \]

\[ P_{12} = \frac{P(X_{k+1} = 2, X_k = 1)}{P(X_k = 1)} = \frac{4p^7q^3}{4p^4q^2} = p^2q^2 \]

\[ P_{20} = \frac{P(X_{k+1} = 0, X_k = 2)}{P(X_k = 2)} = \frac{2p^6q^2}{2p^3q^2} = p \]

\[ P_{21} = \frac{P(X_{k+1} = 1, X_k = 2)}{P(X_k = 2)} = \frac{2p^6q^2}{2p^3q^2} = 2p^3q \]

\[ P_{22} = \frac{P(X_{k+1} = 2, X_k = 2)}{P(X_k = 2)} = \frac{4p^7q^4}{2p^3q^2} = p^2q^2 \]

\[ \square \]

**References**


Reliability of connected (1, 1, 2) or (1, 2, 1) or (2, 1, 1)-out-of-(n, 2, 2): $F$ lattice systems


