ALQUDS OPEN UNIVERSITY جامعة القدس المفتوحة

Solutions Manual for Introductory Linear Algebra

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Introduction

The SolutionsManual for Introductory Linear Algebra has been preparedto be complementary to the textbook to help students to understand the scientific material.

In this Manual there are many exercises and solutions on each section of each unit.As well as the glossary has been prepared in both Arabic and English languages in addition to the table of contents, which also shows therequired material from the textbook.

In addition to this manual and the textbook there is a practical Manualfor Mathcad and QSBprograms.

Dr.Mahmoud Al-Beik

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Textbook: Introductory Linear Algebra with Applications.

Contents

(Required Material from The Textbook)

Chapter 1

Linear Equations and Matrices

1.1.Exercises:

1.Solve the given linear systems by the method of elimination:

(a). x-2y=8

3x+4y=4

(b). 3x+2y+z=2

4x+2y+2z =8

x -y+z =4

(c). 2x+3y=13

x-2y =3

5x+2y =27

2. An oil refinery processes low-sulfur and high-sulfur fuel.Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant.Each ton of high –sulfer fuel requires 4 minutes in the plending plant and 2 minutes in the refining plant .If the plending plant is available for 3 houresand the refining plant is available for 2 houres ,how many tons of each typ of fuel should be manufactured?.

3.A dietician is preparing a meal consisting of foods A,B , and C.Each ounce of foodA contains 2 units of protien, 3 units offat , and 4 units of carbohydrate.Each unit of food B contains 3units of protein ,2 units of fat , and 1 unit of carbohydrate. Each unit of food C contains 3 units of protein ,3 units of fat , and 2 unit of carbohydrate.If the meal must provide exactly 25 units of protein ,24 units of fat , and 21 units of carbohydrate,how many ounces of each type of food should be used?.

1.2.Exercises:

1. Let

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\end{bmatrix} \quad , \quad B=\begin{bmatrix}1 & 0\\ 2 & 1\\ 3 & 2\end{bmatrix} \quad and \quad C=\begin{bmatrix}3 & -1 & 3\\ 4 & 1 & 5\\ 2 & 1 & 3\end{bmatrix}$$

If possible ,compute: B^TC+A

2.If $\begin{bmatrix}a+2b & 2a-b\\ 2c+d & c-2d\end{bmatrix}=\begin{bmatrix}4 & -2\\ 4 & -3\end{bmatrix}, find \quad a,b,c \quad and \quad d$

3. Consider the following linear system:

$$\begin{aligned}&2x+w=7\\&3x+2y+3z=-2\\&2x+3y-4z=3\\&x+3z=5\end{aligned}$$

(a) Find the coefficient matrix.
(b) Write the linear system in matrix form.
(c) Find the augmented matrix.

4. Write the linear system with augmented matrix?

$$\begin{bmatrix}-2 & -1 & 0 & 4 & : & 5\\ -3 & 2 & 7 & 8 & : & 3\\ 1 & 0 & 0 & 2 & : & 4\\ 3 & 0 & 1 & 3 & : & 6\end{bmatrix}$$

5.A manufacturer makes two kinds of products,P and Q,at each of two plants,X and Y.In making these products , the pollutants sulfurdioxide,nitric oxide,and particulate matter are produced.The amounts of pollutants produced are given (in kilograms)by the matrix

SulfurNitricParticulate

Dioxideoxidematter

$$A=\begin{bmatrix}300 & 100 & 150\\200 & 250 & 400\end{bmatrix}\begin{matrix}\text{Pr}oductP\\ \text{Pr}oductQ\end{matrix}$$

Stateand federal ordinances require that these pollutantsbe removed.The daily cost of removing each kilogram of pollutant is given (in dollars)by the matrix

Plant XPlant Y

$$B=\begin{bmatrix}8 & 12\\7 & 9\\15 & 10\end{bmatrix}\begin{matrix}Sulfur \quad dioxide\\ Nitric \quad Oxide\\ Particulate \quad matter\end{matrix}$$

What do the entries in the matrix ProductAB tell the manufacturer?

6. A photography business has a store in each of the following cities: New York,Denver,and Los Angeles.AParicular make of camera is available in automatic,semiautomic,andnonautomaticmodels.Moreover,each camera has a matched flash unit and a camera is usually sold together with the corresponding flash unit.The selling prices of the cameras and flash units are given (in dollars)by the matrix

AutomaticsemiautomaticNonautomatic

$$A=\begin{bmatrix}200 & 150 & 120\\50 & 40 & 25\end{bmatrix}\begin{matrix}Camera\\ Flash \quad unit\end{matrix}$$

The number of sets (camera and flashunit)available at each store is given by the matrix:

New York Denver Los Angeles

$$B = \begin{bmatrix} 220 & 180 & 100 \\ 300 & 250 & 120 \\ 120 & 320 & 250 \end{bmatrix} \begin{matrix} Automatic \\ Semiautomatic \\ Noautomatic \end{matrix}$$

(a) What is the total value of the Camera in New York?

(b) What is the total value of the flash units in Los Angeles?

1.3.Exercises:

1.If $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, show that **AB=0**.

2.If $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$, *and* $C = \begin{bmatrix} -4 & -3 \\ 0 & -4 \end{bmatrix}$,

show that **AB = AC.**

3. Consider two quick food companies,**M** and **N**.Each year ,company**M** keeps $\frac{1}{3}$ of its customers,while $\frac{2}{3}$ switch to **N**.Each year ,**N** keeps $\frac{1}{2}$ of its customers,while $\frac{1}{2}$ switch to **M**.Suppose that the initial distribution of the market is given by $X_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

(a) Find the distribution of the market after 1 year.

(b) Find the stable distribution of the market.

4.Let **A** and **B** be symmetric matrices.

(a) Show **that A+B** is symmetric.

(b) Show that **AB** is symmetric if and only if**AB=BA**

5. If A is annxn matrix ,prove that

(a)$\mathbf{AA^T}$and $\mathbf{A^T.A}$are symmetric.

(b) $\mathbf{A+A^T}$ is symmetric.

(c)$\mathbf{A - A^T}$ is skew symmetric.

1.4.Exercises:

1.Which of the following matrices are in reduced row echelon form?

$$A=\begin{bmatrix}1&0&0&0&-3\\0&0&1&0&4\\0&0&0&1&2\end{bmatrix} B=\begin{bmatrix}0&1&0&0&5\\0&0&1&0&-4\\0&0&0&-1&3\end{bmatrix}$$

$$C=\begin{bmatrix}0&1&0&0&5\\0&0&1&0&4\\0&1&0&-2&3\end{bmatrix} D=\begin{bmatrix}0&1&0&0&2\\0&0&0&0&-1\\0&0&0&1&4\\0&0&0&0&0\\0&0&0&0&1\end{bmatrix}$$

$$E=\begin{bmatrix}1&0&0&0&2\\0&0&1&0&0\\0&0&0&1&3\\0&0&0&0&0\end{bmatrix} F=\begin{bmatrix}0&0&0&0&0\\0&0&1&2&-3\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$$G=\begin{bmatrix}1&0&0&0&1\\0&1&0&0&2\\0&0&0&1&-1\\0&0&0&0&0\end{bmatrix} H=\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&0&-1\\0&0&0&0\end{bmatrix}$$

2. Let

$$A=\begin{bmatrix}1&0&3\\-3&1&4\\4&2&2\\5&-1&5\end{bmatrix}$$

Find the matrices obtained by performing the followingelements row operations on **A**.

(a) Interchanging the **second and fourth** rows.

(b) Multiplying **the third row** by **3**.

(c) Adding **-3 times** the first row to the fourth row .

3. If

$$A=\begin{bmatrix}1&-2&0&2\\2&-3&-1&5\\1&3&2&5\\1&1&0&2\end{bmatrix}$$

Find a matrix **C**in reduced row echelon form that is row equivalent to **A.**

4. Find all solutions to the given linear system:

$$\begin{aligned}x+y+z&=1\\x+y-2z&=3\\2x+y+z&=2\end{aligned}$$

5. Find all values of **a** for which the resulting of the following linear systemhas:

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2 - 5)z = a$$

(1) No solution,

(2)A unique solution,

(3) Infinity many solutions

6. Solve the linear system with thefollowing given augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 1 & 1 & 0 & : & 3 \\ 0 & 1 & 1 & : & 1 \end{bmatrix}$$

7. A furniture manufacturer makes chairs, coffee tables, and dining-room tables. Each chair requires 10 minutes of sanding, 6 minutes of staining, and 12 minutes of varnishing. Each coffee table requires 12 minutes of sanding, 8 minutesof staining,and 12 minutes of varnishing.Each dining room table requires 15 minutes of sanding, 12 minutesof staining,and 18 minutes of varnishing. The sanding bench is available 16 hours per week, the staining bench 11 hours per week, and the varnishing bench 18 hours per week. How many (per week) of each type of furniture should be made?

1.5. Exercises:

1. Show that

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}$$

is non-singular.

2.Show that

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

is singular.

3.Find the inverse of the following matrix, if possible

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

4. Which of the following linear systems have a nontrivial solution?

(a)

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2y + 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

(b)

$$\begin{aligned} 2x + y - z &= 0 \\ x - 2y - 3z &= 0 \\ -3x - y + 2z &= 0 \end{aligned}$$

5.Consider an industrial process whose matrix is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

Find the input matrix for the following output matrix

$$B = \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$$

6.For what values of λ does the homogeneous system

$$(\lambda - 1).x + 2y = 0$$
$$2x + (\lambda - 1)y = 0$$

have a nontrivial solution?

7. Suppose that **A** and **B** are square matrices and **AB=0**.If **B** is non-singular, find **A**.

8. Let **A** be nxnmatrix.Prove that if **A** is non-singular, then the homogeneous system **AX=O** has trivial solution.

1.6 :Answers to Exercises:

1.1. Exercises:

1. (a)

$x-2y=8 \quad /.2 \quad$ (*Means Multiply the equation by* 2)

$+ \quad$ (*Then add to the second equation*)

$3x+4y=4$

……………………

$5x=20 \Rightarrow x=4$

$$y=\frac{x-8}{2}=\frac{4-8}{2}=-2$$

$\therefore$ *The solution is* $: x=4, y=-2$

b)

$3x+2y+z=2 \quad /.-1$

$+$

$4x+2y+2z=8$

$x+z=6$………. ………. ……(1)

$3x+2y+z=2$

$+$

$x-y+z=4 \quad /.2$

$5x+3z=10$………. …..(2)

$x+z=6 \quad /.-3$

$+$

$5x+3z=10$

$2x=-8 \Rightarrow x=-4 \Rightarrow z=6-x=6-(-4)=10$

$y=x+z-4=-4+10-4=2$

$\therefore \quad x=-4, \quad y=2, \quad z=10$

c)

$2x+3y=13$

$+$

$x-2y=3 \qquad /-2$

$7y=7 \Rightarrow y=1 \Rightarrow x=3+2y=5$

$5(5)+2(1)=27$

$\therefore x=5, y=1$

2. **Suppose that:** x_1 **: low- sulfur,** x_2 **: high- sulfur, and note that 2 hours = 2x60=120 minutes, and 3 hours = 3x60=180 minutes.**

$$5x_1+4x_2=180$$
$$4x_1+2x_2=120$$

$$5x_1 + 4x_2 = 180$$
$$4x_1 + 2x_2 = 120 / -2$$

$$-3x_1 = -60 \Rightarrow x_1 = 20ton$$

$$x_2 = \frac{1}{2}(120 - 4x_1) = \frac{1}{2}(120 - 4.(20)) = 20ton$$

3.Suppose that: x_1 **: food A,** x_2 **: food B ,** x_3 **: food C**

$$2x_1 + 3x_2 + 3x_3 = 25$$
$$3x_1 + 2x_2 + 3x_3 = 24$$
$$4x_1 + x_2 + 2x_3 = 21$$

$$2x_1 + 3x_2 + 3z_2 = 25 \quad /.-2$$
$$+$$
$$4x_1 + x_2 + 2x_3 = 21$$

$$-5x_2 - 4x_3 = -29$$

$$-5x_2 - 4x_3 = -29 \quad /.(-1)$$
$$+$$
$$-5x_2 - 3x_3 = -27$$

$$x_3 = 2 \Rightarrow x_2 = \frac{(27 - 3x_3)}{5} = \frac{21}{5} = 4.2 \Rightarrow x_1 = \frac{25 - 3x_2 - 3x_3}{2} = 3.2$$

$$\therefore \quad x_1 = 3.2, \quad x_2 = 4.2, \quad x_3 = 2$$

1.2.Exercises:

1. $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad B^T.C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} . \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 4 & 22 \\ 8 & 3 & 11 \end{bmatrix}$

$$B^T.C + A = \begin{bmatrix} 18 & 6 & 25 \\ 10 & 4 & 15 \end{bmatrix}$$

2.

$$a+2b=4 \qquad /.-2$$

$$+$$

$$2a-b=-2$$

$$-5b=-10 \Rightarrow b=2, \Rightarrow a=4-2b=0$$

$$2c+d=4 \qquad /.2$$

$$+$$

$$c-2d=-3$$

$$5c=5 \Rightarrow c=1 \Rightarrow d=4-2c=2$$

$$\therefore a=0, b=2, c=1, d=2$$

3.(a): The coefficient matrix is:

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 \\ 2 & 3 & -4 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

(b): Thelinear system in matrix form:

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 \\ 2 & 3 & -4 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$$

C :The augmented matrix.

$$\begin{bmatrix} 2 & 0 & 0 & 1: & 7 \\ 3 & 2 & 3 & 0: & -2 \\ 2 & 3 & -4 & 0: & 3 \\ 1 & 0 & 3 & 0: & 5 \end{bmatrix}$$

4.

$$-2x_1 - x_2 + 4x_4 = 5$$
$$-3x_1 + 2x_2 + 7x_3 + 8x_4 = 3$$
$$x_1 + 2x_4 = 4$$
$$3x_1 + x_3 + 3x_4 = 6$$

5.For each product P or Q, the daily cost of pollution control at plant X or at Plant Y.

6. (a)

$$A = \begin{bmatrix} 200 & 150 & 120 \end{bmatrix} . \begin{bmatrix} 220 \\ 300 \\ 120 \end{bmatrix} = [\$103400]$$

(b)

$$A = \begin{bmatrix} 50 & 40 & 25 \end{bmatrix} . \begin{bmatrix} 100 \\ 120 \\ 250 \end{bmatrix} = [\$16050]$$

1.3. Exercises:

1. $A.B = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} . \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.

$$A.B=\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}8 & -6\\ -8 & 6\end{bmatrix}$$

$$A.C=\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}.\begin{bmatrix}-4 & -3\\ 0 & -4\end{bmatrix}=\begin{bmatrix}8 & -6\\ -8 & 6\end{bmatrix}\Rightarrow AB=AC$$

3. The information can bedisplayed in matrix form as:

$$A=\begin{bmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{2}{3} & \frac{1}{2}\end{bmatrix}$$

(a) The distribution of the market after 1 year is:

$$X_1=A.X_o=\begin{bmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{2}{3} & \frac{1}{2}\end{bmatrix}.\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{4}{9}\\ \frac{5}{9}\end{bmatrix}$$

(b) The stable distribution of the market is $\begin{bmatrix}a\\ b\end{bmatrix}$: a+b=1

$$A=\begin{bmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{2}{3} & \frac{1}{2}\end{bmatrix}.\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix}\Rightarrow\frac{1}{3}a+\frac{1}{2}b=a, \quad , \quad \frac{2}{3}a+\frac{1}{2}b=b$$

$$a=1-b\Rightarrow\frac{1}{3}(1-b)+\frac{1}{2}b=1-b\Rightarrow b=\frac{4}{7}, \quad a=\frac{3}{7}$$

4.

(a): (A+B) is symmetric matrix if:

$(A+B)=(A+B)^T$

$(A+B)^T = A^T + B^T$

But $A=A^T$ and $B = B^T$ because A and B are symmetric matrices , so $(A+B)^T = A^T + B^T = A+B$

(b):- Suppose that AB=BA.

(AB) is symmetric matrix if $(AB)=(AB)^T$

$(AB)^T=B^T . A^T=B.A=AB$

$\therefore (AB)$ *is symmetric matrix*

- Suppose that(AB) is symmetric matrix .

Take $(AB)=(AB)^T=B^T.A^T=B.A$ as both A and B are symmetric matrices.

$\therefore \quad AB = BA$

5.

a. $\mathbf{AA^T}$ issymmetric if $(\mathbf{AA^T})=(A.A^T)^T$

$(A.A^T)^T= (A^T)^T.A^T=A.A^T$

Similarly for $A^T.A$.

b.$\mathbf{A+A^T}$ is symmetric if $\mathbf{A+A^T}=(A+A^T)^T$

$(A+A^T)^T = (A^T)^T+A^T= A+A^T$

c. $\mathbf{A - A^T}$is skew symmetric if $\mathbf{A - A^T} = - (A-A^T)^T$

$- (A-A^T)^T= -(A^T-(A^T)^T)=-A^T+A=A-A^T$

1.4.Exercises:

1. A,E,G

2.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 5 & -1 & 5 \\ 4 & 2 & 2 \\ -3 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 12 & 6 & 6 \\ 5 & -1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 2 & -1 & 4 \end{bmatrix}$

3.

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -3 & -1 & 5 \\ 1 & 3 & 2 & 5 \\ 1 & 1 & 0 & 2 \end{bmatrix} \cong \begin{matrix} \\ -2R_1+R_2 \to \\ -R_1+R_3 \to \\ -R_1+R_4 \to \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & 2 & 3 \\ 0 & 3 & 0 & 0 \end{bmatrix} \cong \begin{matrix} 2R_2+R_1 \\ \\ -5R_2+R_3 \to \\ -3R_2+R_4 \to \end{matrix} \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 3 & -3 \end{bmatrix} \cong \ldots\ldots\ldots \cong$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 1 & -2 & : & 3 \\ 2 & 1 & 1 & : & 2 \end{bmatrix} \cong \begin{matrix} \\ -R_1+R_2 \to \\ -2R_2+R_3 \to \end{matrix} \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 0 & -3 & : & 2 \\ 0 & -1 & -1 & : & 0 \end{bmatrix} \cong \begin{matrix} R_3+R_2 \to \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 0 & -3 & : & 2 \\ 0 & -1 & -1 & : & 0 \end{bmatrix} \cong$$

$$R_3 \Leftrightarrow R_2 \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & -1 & : & 0 \\ 0 & 0 & -3 & : & 2 \end{bmatrix} \cong \begin{matrix} \\ -R_2 \\ -\frac{1}{3}R_3 \to \end{matrix} \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 1 & : & -\frac{2}{3} \end{bmatrix} \cong -R_3+R_2 \to \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & \frac{2}{3} \\ 0 & 0 & 1 & : & -\frac{2}{3} \end{bmatrix}$$

$x_{1=1}$, $x_2 = 0.667$, $x_3 = -0.667$,

5.

$$\begin{bmatrix} 1 & 1 & -1 & : & 2 \\ 1 & 2 & 1 & : & 3 \\ 1 & 1 & a^2-5 & : & a \end{bmatrix} \cong -R_1+R_2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & : & 2 \\ 1 & 2 & 1 & : & 3 \\ 0 & 0 & a^2-4 & : & a-2 \end{bmatrix}$$

(1) No solution:

$$a^2-4=0, a-2 \neq 0 \Rightarrow a=-2$$

(b)**)** A unique solution:

$$a^2-4 \neq 0, \Rightarrow a \in R/\{2,-2\}$$

(c) Infinity many solutions:

$$a^2-4=0, a-2=0 \Rightarrow a=2$$

(6)

$$\begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 1 & 1 & 0 & : & 3 \\ 0 & 1 & 1 & : & 1 \end{bmatrix} \cong -R_1+R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 0 & 0 & -1 & : & 3 \\ 0 & 1 & 1 & : & 1 \end{bmatrix} \cong -R_3+R_1 \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 0 & -1 & : & 3 \\ 0 & 1 & 1 & : & 1 \end{bmatrix} \cong$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 1 & : & 1 \\ 0 & 0 & -1 & : & 3 \end{bmatrix} \cong R_3+R_2 \rightarrow \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 4 \\ 0 & 0 & -1 & : & 3 \end{bmatrix} \cong -R_3 \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 4 \\ 0 & 0 & 1 & : & -3 \end{bmatrix}$$

$x_1=-1$, $x_2=4$, $x_3=-3$

(7)

Let: x=number of chairs

y=number of coffee tables

z= number of dining room tables.

$$\begin{aligned} 10x+12y+15z &= 960 \\ 6x+8y+12z &= 660 \\ 12x+12y+18z &= 1080 \end{aligned}$$

$$\begin{bmatrix} 10 & 12 & 15 & : & 960 \\ 6 & 8 & 12 & & 660 \\ 12 & 12 & 18 & : & 1080 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & : & 30 \\ 0 & 1 & 0 & : & 30 \\ 0 & 0 & 1 & : & 20 \end{bmatrix} \Rightarrow$$

x=30 , y = 30 z = 20

1.5. Exercises:

1.A is non-singular matrix because there exist matrix B such that A.B=B.A=I

$$\begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} . \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} . \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Suppose that

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Is an inverse of A. Then

$$A.B = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ -4a-2c & -4b-2d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding elements of these two matrices, we obtained the linear system

$2a + c = 1 \quad /.2$

$+$

$-4a - 2c = 0$

$2 = 0 \Rightarrow$

So this linear system has no solution .Hence there is no such matrix B and A is singular matrix.

3.

$$C^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

4.

$$(a) \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cong \underset{-R_1+R_2 \to}{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Singular \quad Matrix$$

$$\therefore \quad The \quad system \quad has \quad nontrivial \quad solution$$

$$(b) \quad \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & -3 \\ -3 & -1 & 2 \end{bmatrix} \cong R_1 \Leftrightarrow R_2 \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{bmatrix} \cong \begin{matrix} -2R_1+R_2 \to \\ 3R_1+R_3 \end{matrix} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 5 & 5 \\ 0 & -7 & -7 \end{bmatrix} \cong$$

$$\cong \frac{7}{5}R_2 + R_3 \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Singular \quad Matrix$$

$$\therefore \quad The \quad system \quad has \quad nontrivial \quad solution$$

5. X= A^{-1}.B

$$A^{-1} = \begin{bmatrix} -1.5 & -1 & 3.5 \\ 2.5 & 2 & -5.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}$$

$$X = A^{-1}.B = \begin{bmatrix} -1.5 & -1 & 3.5 \\ 2.5 & 2 & -5.5 \\ 0.5 & 0 & -0.5 \end{bmatrix} . \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} -30 \\ 60 \\ 10 \end{bmatrix}$$

$$\lambda = -1, \quad \lambda = 3$$

6.

$$\begin{bmatrix} \lambda-1 & 2 & : & 0 \\ 2 & \lambda-1 & : & 0 \end{bmatrix} \cong \begin{matrix} -2R_1 \to \\ (\lambda-1) \to \end{matrix} \begin{bmatrix} -2\lambda+2 & -4 & : & 0 \\ 2\lambda-2 & (\lambda-1)^2 & : & 0 \end{bmatrix} \cong \begin{bmatrix} -2\lambda+2 & -4 & : & 0 \\ 0 & \lambda^2-2\lambda-3 & : & 0 \end{bmatrix} \Rightarrow$$

$$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow \lambda = 3, \lambda = -1$$

7. $A.B = O \Rightarrow A.B.B^{-1} = O.B^{-1} = O \Rightarrow A = O$

8. $A.X = O \Rightarrow A^{-1}.A.X = O \Rightarrow X = O$

1.7Glossary:

Matrix:مصفوفة

Zero matrix: مصفوفة صفرية

Square Matrix: مصفوفة مربعة

Reduced Row Echelon form:الشكل الصفي المميز

Diagonal Matrix:المصفوفة القطرية

Identity Matrix: مصفوفة الوحدة

Triangular Matrix:مصفوفة مثلثية

Coefficient Matrix:مصفوفة المعاملات

Augmented Matrix:المصفوفة الممتدة

Linear System:النظام الخطي

Inverse of a Matrix:النظير الضربي للمصفوفة (معكوس المصفوفة)

Transpose of a Matrix:منقول المصفوفة

Rank of a Matrix:رتبة المصفوفة

Elementary Row Operations: عمليات الصف البسيط

Chapter 2

Determinants

2.1. Exercises:

1. Find the number of inversions in each of following permutations of

$s = \{1,2,3,4,5\}$

(a) 52134
(b) 45213
(c) 42135

2.Determinate whether each of the following permutations of

$s = \{1,2,3,4\}$ Is even or odd

(a) 4213
(b) 1243
(c) 1234

3. Compute thefollowing determinants:

(a) $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$

4. If

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -4$$

Find the determinants of the following matrices:

$$B = \begin{bmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{bmatrix}$$

$$C = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 2c_1 & 2c_2 & 2c_3 \end{bmatrix}$$

And

$$D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 + 4c_1 & b_2 + 4c_2 & b_3 + 4c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

5. Evaluate the following determinants using the properties of this Section:

(a) $$A = \begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

(b) $$B = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$$

(c) $$C = \begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$$

6. If A and B are 3x3 matrices with $|A| = 2 \quad and \quad |B| = 4,$

Calculate

(a) $|3A|$

(b) $|A(3B)^{-1}|$

(c) $|3AB^{-1}|$

(d) $|A^{-1}.B^{T}|$

(e) $|A^{2}|$

7. Show that if $A=A^{-1}$, then $|A| = \pm 1$

8.Show that if $A^{T}=A$, then $|A| = \pm 1$

9. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

This determinant is called a Vandermonde determinant

2.2.Exercises:

1. Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$$

Compute all the cofactors.

2. Compute the determinants of the following matrices:

(a) $$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

(a) Find adj A.

(b) Compute $|A|$

(c) Compute A.(adj A)

4. Compute the inverse of the following matrix:

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

5.Use theorem 2.12 to determine which of the following matrices are non-singular:

(a) $$A = \begin{bmatrix} 4 & 3 & -5 \\ -2 & -1 & 3 \\ 4 & 6 & -2 \end{bmatrix}$$

(b) $$B = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & -6 & 4 & 1 \\ 3 & 5 & -1 & 3 \\ 4 & -6 & 5 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 5 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

(c)

(d) $D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

6. Use Corollary 2.3 to find out whether the following homogeneous systems have nontrivial solutions:

(a) x+y-z=0

2x+y+2z=0

3x-y+z=0

(b) x+y+2z+w=0

2x-y+z-w=0

3x+y+2z+3w=0

2x-y-z+w=0

7. Solve the following linear system by Cramer's rule**:**

X+y+z-2w=-4

2y+z+3w=4

2x+y-z+2w=5

x-y+w=4

8. Prove that $|adjA| = |A|^{n-1}$

2.3Answers to Exercises:

2.1.Exercises:

1. (a) five inversions: 52,51,53,54,21.

   (b) Seven inversions: 42,41,43,52,51,53,21

   (c) Fourinversions : 42,41,43,21.

2. (a) even(n=4: 42,41,43,21)

   (b) Odd (n=1: 43)

   (c) Even(n=0)

3. (a) (2).(2)-(-1)(3)=7

   (b) (4)(-2)(3)=-24

4. $|B| = -1.|A| = 4$

   $|C| = 2.|A| = -8$

   $|D| = |A| = -4$

5. (a)

$$\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \begin{matrix} \\ \frac{-5}{4}R_1 + R_2 \to \\ \frac{-1}{2}R_1 + R_3 \to \end{matrix} \begin{vmatrix} 4 & -3 & 5 \\ 0 & \frac{23}{4} & \frac{-25}{4} \\ 0 & \frac{3}{2} & \frac{3}{2} \end{vmatrix} = (\frac{-3}{2})(\frac{4}{23})R_2 + R_3 \to \begin{vmatrix} 4 & -3 & 5 \\ 0 & \frac{23}{4} & \frac{-25}{4} \\ 0 & 0 & \frac{144}{46} \end{vmatrix} \Rightarrow$$

$$|A| = 4(\frac{23}{4})(\frac{144}{46}) = 72$$

(b) $|B| = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 3 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} = -R_4 + R_2 \to \begin{vmatrix} 2 & 0 & 1 & 4 \\ -8 & -6 & 0 & -8 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} =$

$|B| = \quad 2R_3 + R_2 \to \begin{vmatrix} 2 & 0 & 1 & 4 \\ -4 & 0 & -2 & -8 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} = \frac{1}{2}R_2 + R_1 \to \begin{vmatrix} 0 & 0 & 0 & 0 \\ -4 & 0 & -2 & -8 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} \Rightarrow$

$|B| = 0$

(c)

$$|C| = \begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix} = 4(2)(-3) = -24$$

6. (a) $|3A| = 3^3.|A| = 54$

(b) $|A(3B)^{-1}| = |A|.\frac{1}{|3B|} = |A|.\frac{1}{3^3.|B|} = 2.\frac{1}{27(4)} = \frac{1}{54}$

(c) $\left|3AB^{-1}\right| = 3^3.|A|.\frac{1}{|B|} = 27.(2).\frac{1}{4} = \frac{27}{2}$

(d) $\left|A^{-1}.B^T\right| = \left|A^{-1}\right|.\left|B^T\right| = \frac{1}{|A|}.|B| = \frac{4}{2} = 2$

(e) $\left|A^2\right| = |A|.|A| = 4$

7. $A = A^{-1} \Rightarrow |A| = \left|A^{-1}\right| = \frac{1}{|A|} \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

8. $A^T = A^{-1} \Rightarrow \left|A^T\right| = |A| = \left|A^{-1}\right| = \frac{1}{|A|} \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

9. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{matrix} \\ -R_1 + R_2 \to \\ -R_1 + R_3 \to \end{matrix} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & c+a \end{vmatrix}$$

$$(b-a)(c-a)\begin{vmatrix} 1 & a & a^1 \\ 0 & 1 & a+b \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$$

2.2.Exercises:

1.

$A_{11} = (-1)^{1+1}.\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -11,$ $A_{12} = (-1)^{1+2}.\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix} = 29$

$A_{13} = (-1)^{1+3}.\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1,$ $A_{21} = (-1)^{2+1}.\begin{vmatrix} 0 & -2 \\ 2 & -3 \end{vmatrix} = -4$

$A_{22} = (-1)^{2+2}.\begin{vmatrix} 1 & -2 \\ 5 & -3 \end{vmatrix} = 7,$ $A_{23} = (-1)^{2+3}.\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2$

$$A_{31}=(-1)^{3+1}.\begin{vmatrix}0 & -2\\ 1 & 4\end{vmatrix}=2, \qquad A_{32}=(-1)^{3+2}.\begin{vmatrix}1 & -2\\ 3 & 4\end{vmatrix}=-10$$

$$A_{33}=(-1)^{3+3}.\begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}=1$$

2. (a) $$|A|=1.\begin{vmatrix}5 & 2\\ 2 & 0\end{vmatrix}-2\begin{vmatrix}-1 & 2\\ 3 & 0\end{vmatrix}+3\begin{vmatrix}-1 & 5\\ 3 & 2\end{vmatrix}=-43$$

(b) $$|B|=-3.\begin{vmatrix}4 & 2 & 1\\ 1 & 0 & 3\\ 2 & 3 & 4\end{vmatrix}-2\begin{vmatrix}4 & -4 & 1\\ 1 & 2 & 3\\ 2 & 0 & 4\end{vmatrix}+1.\begin{vmatrix}4 & -4 & 2\\ 1 & 2 & 0\\ 2 & 0 & 3\end{vmatrix}=75$$

3.

(a) $$A_{11}=(-1)^{1+1}.\begin{vmatrix}4 & 1\\ -4 & 5\end{vmatrix}=24, \qquad A_{12}=(-1)^{1+2}.\begin{vmatrix}-3 & 1\\ 4 & 5\end{vmatrix}=19$$

$$A_{13}=(-1)^{1+3}.\begin{vmatrix}-3 & 4\\ 4 & -4\end{vmatrix}=-4 \qquad A_{21}=(-1)^{2+1}.\begin{vmatrix}2 & 8\\ -4 & 5\end{vmatrix}=-42$$

$$A_{22}=(-1)^{2+2}.\begin{vmatrix}6 & 8\\ 4 & 5\end{vmatrix}=-2 \qquad A_{23}=(-1)^{2+3}.\begin{vmatrix}6 & 2\\ 4 & -4\end{vmatrix}=32$$

$$A_{31}=(-1)^{3+1}.\begin{vmatrix}2 & 8\\ 4 & 1\end{vmatrix}=-30 \qquad A_{32}=(-1)^{3+2}.\begin{vmatrix}6 & 8\\ -3 & 1\end{vmatrix}=-30$$

$$A_{33} = (-1)^{3+3}.\begin{vmatrix} 6 & 2 \\ -3 & 4 \end{vmatrix} = 30 \Rightarrow C = \begin{bmatrix} 24 & 19 & -4 \\ -42 & -2 & 32 \\ -30 & -30 & 30 \end{bmatrix}$$

$$Adj(A) = C^T = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix}$$

(b) $$|A| = (6)(24) + (2)(19) + 8(-4) = 150$$

(c) $$A.Adj(A) = |A|.I_3 = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix}$$

4. $$A_{11} = \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = -10, \quad A_{12} = -1\begin{vmatrix} 0 & 4 \\ 0 & -2 \end{vmatrix} = 0$$

$$A_{13} = \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} = 0, \quad A_{21} = -1\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} = 2$$

$$A_{22} = \begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} = -8, \quad A_{23} = -1\begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} = -4$$

$$A_{31} = \begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix} = -6, \quad A_{32} = -1\begin{vmatrix} 4 & 2 \\ 0 & 4 \end{vmatrix} = -16$$

$$A_{33} = \begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} = 12 \Rightarrow C = \begin{bmatrix} -10 & 0 & 0 \\ 2 & -8 & -4 \\ -6 & -16 & 12 \end{bmatrix} \Rightarrow |A| = (-10).(4) = -40$$

$$Adj(A) = C^T = \begin{bmatrix} -10 & 2 & -6 \\ 0 & -8 & -16 \\ 0 & -4 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}.Adj(A) = \begin{bmatrix} \frac{1}{4} & \frac{-1}{20} & \frac{3}{20} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{1}{10} & \frac{-3}{10} \end{bmatrix}$$

5. $(a) \quad |A| = 0 \Rightarrow Singular \quad Matrix$

$(b) \quad |B| = 0 \Rightarrow Singular \quad Matrix$

$(c) \quad |C| = 0 \Rightarrow Singular \quad Matrix$

$(d) \quad |D| = -2 \Rightarrow Nonsingular \quad Matrix$

6.

$$(a) \quad \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 4 \Rightarrow Has \quad Nontrivial \quad Solution$$

$$(b)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & -1 \\ 3 & 1 & 2 & 3 \\ 2 & -1 & -1 & 1 \end{vmatrix} = 0 \Rightarrow Has \quad Trivial \quad Solution.$$

7.

$$A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 4 \\ 5 \\ 4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -4 & 1 & 1 & -2 \\ 4 & 2 & 1 & 3 \\ 5 & 1 & -1 & 2 \\ 4 & -1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 4 & 1 & 3 \\ 2 & 5 & -1 & 2 \\ 1 & 4 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & -4 & -2 \\ 0 & 2 & 4 & 3 \\ 2 & 1 & 5 & 2 \\ 1 & -1 & 4 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 0 & 4 \end{bmatrix}$$

$$|A| = -30 \quad , \quad |A_1| = -30 \quad , \quad |A_2| = 30 \quad , \quad |A_3| = 0$$

$$|A_4| = -60$$

$$x = \frac{|A_1|}{|A|} = 1, \quad y = \frac{|A_2|}{|A|} = -1, \quad z = \frac{|A_3|}{|A|} = 0, \quad w = \frac{|A_4|}{|A|} = 2$$

8.

$$A.Adj(A) = |A|.I_n \Rightarrow |A.Adj(A)| = \left||A|.I\right| \Rightarrow$$

$$|A|.|Adj(A)| = |A|^n.|I| \Rightarrow |Adj(A)| = \frac{|A|^n}{|A|} = |A|^{n-1}$$

2.4Glossary:

The Minor of a_{ij} :a_{ij} المحدد المتمم للعنصر

Cofactor:المتعامل

Determinant: المحدد

Adjoint of a Matrix: قرين المصفوفة

Cramer's Rule :قاعدة كرايمر

Chapter 3

Vectors and Vector Spaces

3.1.Exercises:

1. Determine the head of the vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ whose tail is at (3,2).

2.Determine the tail of the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ whose head is at (1,2).

3.Let X=(1,2) , Y=(-3,4),Z=(x,4),and U=(-2,y).Find x and y so that:

(a) Z=2X

(b) $\frac{3}{2}U = Y$

(c) $Z + U = X$

4. Find the length of the following vectors:

(a) (1,2)

(b)(-3,-4)

5. Find the distance between pairs of points:

(a) (4,2) , (1,2)

(b) (-2,-3), (0,1)

6. Find a unit vector in the direction of X

(a) X=(2,4)

(b) X=(0,-4)

7. Find X.Y.

(a) X=(0,-1),Y=(1,0)

(b) X=(2,2) , Y = (4,-4)

8. Write each of the following vectors in terms of i and j.

(a) (1,3)

(b) (-2,-3)

9. A ship is being pushed by a tugboat with a force of 300 pounds along the negative y-axis while another tugboat is pushing along the negative x-axis with a force of 400 ponds. Find the magnitude and sketch the direction of the resultant force.

3.2.Exercises:

1. Find X+Y,X-Y,2X, and 3X-2Y if X=(1,2,-3),Y=(0,1,-2).

2. Let X=(1,-2,3),Y=(-3,-1,3),Z=(x,-1,y),and U=(3,u,2).Find x,y,and u so that:

(a) Z+Y=X

(b) Z+U=Y

3. Find the length of the following vectors:

(a) (2,3,4)

(b) (0,-1,2,3)

4.Find the distance between the following pairs of points.

(a) (1,1,0),(2,-3,1)

(b) (4,2,-1,6),(4,3,1,5)

5. Find X.Y.

(a) X=(2,3,1),Y=(3,-2,0)

(b) X=(1,2,-1,3),Y=(0,0,-1,-2)

6.Which of the following vectors:

X_1=(4,2,6,-8),X_2=(-2,3,-1,-1),X_3=(-2,-1,-3,4),X_4=(1,0,0,2),X_5=(1,2,3,-4),

X_6=(0,-3,1,0)are:

(a) Orthogonal?
(b) Parallel?
(c) In the same direction?

7. Find a unit vector in the direction of X.

(a) X = (1,2,-1)

(b) X = (1,2,3,4)

8.Write each of the following vectorsin R^3in terms of i,j,and k.

(a) (1,2,-3).

(b) (2,3,-1).

9.Write each of the following vectorsin R^3as a3x1 matrix.

(a) 2i+3j-4k.

(b) i+2j.

10. A large steel manufacturer, who has 2000 employees, lists each employee's salary as a component of a vector **S** in R^{2000}.If an 8 percent across-the-board salary increase has been approved, find an expression involving **S** giving all the new salaries.

3.4.Exercises:

1. Determine whether thegiven **set** together with the given operations is a vector space. If it is not a vector space, list the properties of Definition 1 that fail to hold.

(a) The set of all ordered triples of real numbers (x,y,z)with the operations (x,y,z) and $\oplus(x',y',z')=(x',y+y',z')$

$c.(x,y,z)=(cx,cy,cz)$

(b) The set of all ordered triples of real numbers of the form (0,0,z)with the operations $(0,0,z)\oplus(0,0,z')=(0,0,z+z')and c.(0,0,z)=(0,0,cz)$

(c) The set of all positive real numbers with the operations $x\oplus y=xy$

And $c.x=x^c$

2. Which of the following subsets of R^4 are subspaces of R^4? The set of all vectors of the form

(a) (a,b,c,d),where a-b=2

(b) (a,b,c,d),where c=a+2b and d=a-3b.

(c) (a,b,c,d),where a=0 and b = -d

3. Which of the following subsets of the vector space of all 2x3 matrices under the usual operations of matrix addition and scalar multiplication are subspaces?

(a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}, where \quad b=a+c$

(b) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, where \quad a=-2c \quad and\ f=2e+d.$

3.5.Exercises:

1.Which of the following vectors are linear combinations of X_1=(4,2,-3), X_2=(2,1,-2),and X_3=(-2,-1,0)?

(a) (4,2,-6)

(b) (-2,-1,1)

2.Which of the following vectors are linear combinations of

$P_1(t)=t^2+2t+1$, $p_2(t)=t^2+3$, $p_3(t)=t-1$?

(a) t^2+t+2.

(b) $-t^2+t-4$.

3. Which of the following sets of vectors span R^4?

(a) (1,0,0,1),(0,1,0,0),(1,1,1,1),(1,1,1,0).

(b) (1,2,1,0),(1,1,-1,0),(0,0,0,1).

(c) (6,4,-2,4),(2,0,0,1),(3,2,-1,2),(5,6,-3,2),(0,4,-2,-1).

(d)(1,1,0,0),(1,2,-1,1),(0,0,1,1,),(2,1,2,1)

4. Do the polynomials $t^3+2t+1, t^2-t+2, t^3+2, -t^3+t^2-5t+2$ span P3?

5.Which of the following sets of vectors in R^3 are linearly dependent? For those that are ,express one vector as a linear combination of the rest.

(a) $X_1=(4,2,1), X_2=(2,6,-5), X_3=(1,-2,3)$

(b)) $X_1=(1,2,-1), X_2=(3,2, 5)$

(c) $X_1=(1,1,0), X_2=(0,2,3), X_3=(1, 2,3), X_4=(3,6,6)$

(d) $X_1=(1,2,3), X_2=(1,1,1), X_3=(1,0,1)$

3.6.Exercises:

1. Which of the following vectors are bases for R^2?

(a) (1,3),(1,-1).

(b) (1,2),(2,-3),(3,2)

(c) (1,3),(-2,6)

2. Which of the following vectors are bases for P_3?

(a) t^3+2t^2+3t, $2t^3+1$, $6t^3+8t^2+6t+4$, t^3+2t^2+t+1

(b) t^3+t^2+t+1, t^3+2t^2+t+3, $2t^3+t^2+3t+2$, t^3+t^2+2t+2

3. Let W be the subspace of P_3 spanned by

$$\{t^3+t^2-2t+1, t^2+1, t^3-2t, 2t^3+3t^2-4t+3\}$$

Find a basis for What is the dimension of W ?

4.Find a basis for R^4 that includes the vectors (1,0,1,0),and (0,1,-1,0)

5. Find the dimension of the solution space of

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.7.Exercises:

1. Find a basis for the subspace V of R^4 spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \quad and \quad \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

2. Find the row and column ranks of

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}$$

3. Is

$$S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

a linearly independent set of vectors in R^3?

4. Determine which of the linear systems have a solution by comparing the ranks of the coefficient and augmented matrices

(a) $$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3.8Answers to Exercises:

3.1.Exercises

1. $x=x_1+m=3+(-2)=1$

   $y=y_1+n=2+5=7$

   The head of the vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ is (1,7)

2. $x=x_2-m=1-(2)=-1$

   $y=y_2-n=2-5=-3$

   The tail of the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is (-1,-3)

3. (a) (x,4)=2(1,2)=(2,4) , x=2

   (b) $\frac{3}{2}(-2, y) = (-3,4) \Rightarrow (-3, \frac{3}{2} y) = 4 \Rightarrow 4 = \frac{3}{2} y, \qquad y = \frac{8}{3}$

   (c) $$(x,4) + (-2, y) = (1,2) \Rightarrow (x-2, y+4) = (1,2)$$
   $$x - 2 = 1 \Rightarrow x = 3$$
   $$y + 4 = 2 \Rightarrow y = -2$$

4. (a) $\sqrt{1+2^2} = \sqrt{5}$

   (b) $\sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$

5.(a) $\sqrt{(4-1)^2+(2-2)^2}=3$

(b) $\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$

6. (a) $\hat{X}=\frac{1}{\sqrt{20}}(2,4)$

(b) $\hat{X}=\frac{1}{4}(0,-4)$

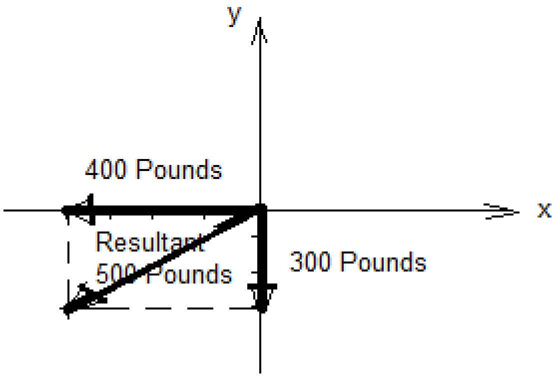
7. (a) X.Y = (0).(1)+(-1).(0)=0

(b) X.Y= (2).(4)+(2).(-4)=0

8. (a) $\hat{i}+3\hat{j}$

(b) $-2\hat{i}-3\hat{j}$

9.



3.2.Exercises:

1. X+Y=(1,2,-3)+(0,1,-2)=(1,3,-5)

X-Y=(1,2,-3)-(0,1,-2)=(1,1,-1)

2X=(2,4,-6)

3X-2Y=3(1,2,-3)-2(0,1,-2)=(3,4,-5)

2\. (a) (x,-1,y)+(-3,-1,3) = (1,-2,3)

(x-3,-2,y+3)=(1,-2,3),

x-3=1 , x = 4

y+3 = 3 , y = 0

(b) (x,-1,y)+(3,u,2)= (-3,-1, 3)

(x+3,u-1,y+2)= (-3,-1,3)

x+3=-3, x=-6

u-1=-1, u=0

y+2=3,y=1

3\. (a) $\sqrt{4+9+16} = \sqrt{29}$

(b) $\sqrt{0+1+4+9} = \sqrt{14}$

4\. (a) $\sqrt{(2-1)^2+(-3-1)^2+(1-0)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

(b) $\sqrt{(4-4)^2+(3-2)^2+(1-(-1))^2+(5-6)^2} = \sqrt{1+4+1} = \sqrt{6}$

5. (a) (2,3,1).(3,-2,0) = 6 + (-6)+0=0

(b) (1,2,-1,3).(0,0,-1,-2) = 0 + 0 + 1+ (-6)= - 5

6. (a) X_1 and X_2 , X_1 and X_6 , X_2 and X_3 , X_3 and X_6 , X_4 and X_6

($X_1.X_2=0$, $X_1.X_6=0$,$X_2.X_3=0$, $X_3.X_6=0$,$X_4.X_6=0$)

(b) X_1 and X_3 (*because* $|X_1.X_3| = \|X_1\|.\|X_3\|$)

(C) None. (*because* $X_{\hat{i}}.X_{\hat{j}} \neq \|X_{\hat{i}}\|.\|X_{\hat{j}}\|$)

$$\hat{X} = \frac{1}{\sqrt{6}}(1,2,-1)$$

7. (a)

(b) $\hat{X} = \frac{1}{\sqrt{30}}(1,2,3,4)$

8. (a) $\hat{i} + 2\hat{j} - 3\hat{k}$

(b) $2\hat{i} + 3\hat{j} - \hat{k}$

9. (a) $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

10. 1.08 . S

3.4.Exercises:

1. (a)

Let X=(x_1,y_1,z_1) , Y = (x_2,y_2,z_2) , Z(x_3,y_3,z_3),c and d in R.

$$X \oplus Y = (x_2, y_1 + y_2, z_2) \neq Y \oplus X = (x_1, y_2 + y_1, z_1)$$

$$X \oplus (Y \oplus Z) = (x_1, y_1, z_1) \oplus ((x_2, y_2, z_2) \oplus (x_3, y_3, z_3)) \neq (X \oplus Y) \oplus Z$$

$$X \oplus (-X) = (x_1, y_1, z_1) + (-x_1, -y_1, z_1) = (-x_1, y_1 - y_1, -z_1) = (-x_1, 0, -z_1) \neq (0,0,0)$$

$$(c+d).X = ((c+d)x_1, (c+d)y_1, (c+d)z_1) \neq c.X \oplus d.X = (d.x_1, cy_1 + dy_1, dz_1)$$

Not a vector Space : (a) , (c) , (d) , and (f) do not hold .

(b) Vector Space.

(c) Vector Space.

2.

(a) Let X=(a_1,b_1,c_1): a_1-b_1=2, Y = (a_2,b_2,c_2): a_2-b_2=2

X+Y=(a_1+a_2,b_1+b_2,c_1+c_2):a_1+a_2-b_1-b_2=4 $\neq 2$

$X+Y \quad \notin W$

c.X=(ca_1,cb_1,cc_1):c(a_1-b_1)=2c $\neq 2$

$c.X \quad \notin W$

So W is not a subspace of R^4 .

(b) is Subspace of R^4.

(c) isSubspace of R^4.

3.(a) Let $X=\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix} : b_1 = a_1 + c_1 \quad \ldots(1)$

$$Y=\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix} : b_2 = a_2 + c_2 \quad \ldots\ldots(2)$$

$$X+Y=\begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & 0 & 0 \end{bmatrix}$$

$$(1)+(2) \Rightarrow b_1+b_2 = a_1+a_2+c_1+c_2 \Rightarrow X+Y \in W$$

$$c.X=\begin{bmatrix} ca_1 & cb_1 & cc_1 \\ cd_1 & 0 & 0 \end{bmatrix}$$

$$(1) \Rightarrow b_1 = a_1 + c_1 \quad /.c \Rightarrow cb_1 = ca_1 + cc_1 \Rightarrow c.X \in W \Rightarrow$$

W is subspace.

(b) Subspace

3.5.Exercises:

1. **(a) -2X_1+6X_2+0.X_3=(4,2,-6)**

(b) -X_1+X_2+0.X_3=(-2,-1,1)

2. (a) $\frac{1}{2}P_1(t)+\frac{1}{2}P_2(t)+0.P_3(t)=t^2+t+2$

(b) $\frac{1}{2}P_1(t)-\frac{3}{2}P_2(t)+0.P_3(t)=-t^2+t-4$

3. (a)**To show that** (1,0,0,1),(0,1,0,0),(1,1,1,1),(1,1,1,0) **spans R^4,we let X=(a,b,c,d) be any vector in R^4 .We now seek constants k_1 , k_2,k_3,and k_4 such that:**

K_1(1,0,0,1)+k_2(0,1,0,0)+k_3(1,1,1,1)+k_4(1,1,1,0)**=(a,b,c,d)**

$$\therefore \quad k_1+k_3+k_4=a$$
$$k_2+k_3+k_4=b$$
$$k_3+k_4=c$$
$$k_1+k_3=d \Rightarrow$$

K_1=d-a+c

K_2=c-a

K_3=b-c

K_4=a-c

So (1,0,0,1),(0,1,0,0),(1,1,1,1),(1,1,1,0) **spans R^4**

(b) **To show that**(1,2,1,0),(1,1,-1,0),(0,0,0,1)**spans R^3,we let X=(a,b,c,d) be any vector in R^4 .We now seek constants k_1 ,k_2 and k_3 such that:**

K_1(1,2,1,0)+k_2(1,1,-1,0)+k_3(0,0,0,1)=(a,b,c,d)

$$\therefore \quad k_1+k_2=a$$
$$2k_1+k_2=b$$
$$k_1-k_2=c$$
$$k_3=d \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & :a \\ 2 & 1 & 0 & 0 & :b \\ 1 & -1 & 0 & 0 & :c \\ 0 & 0 & 1 & 0 & :d \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 & : & b-3a \\ 0 & -1 & 0 & 0 & : & b-2a \\ 0 & 0 & 0 & 0 & ; & 3a-2b+c \\ 0 & 0 & 1 & 0 & ; & d \end{bmatrix} \Rightarrow 3a-2b+c=0$$

$c = 2b - 3a \Rightarrow$ *so the vector* (1,1,1,1) *is not spanned by*

the vectors(1,2,1,0),(1,1,-1,0)(0,0,0,1) $\therefore$ these vectors do not *spans* $R4$.

(c)**To show that** (6,4,-2,4),(2,0,0,1),(3,2,-1,2),(5,6,-3,2),(0,4,-2,-1) **spans R^4,we let X=(a,b,c,d) be any vector in R^4 .We now seek constants k_1 , k_2,k_3,k_4 and k_5 such that:**

$K_1(6,4,-2,4)+k_2(2,0,0,1)+k_3(3,2,-1,2)+k_4(5,6,-3,2)+k_5(0,4,-2,-1)=(a,b,c,d)$

$$\therefore \quad 6k_1 + 2k_2 + 3k_3 + 5k_4 = a$$
$$4k_1 + 2k_3 + 6k_4 + 4k_5 = b$$
$$-2k_1 - k_2 - 3k_4 - 2k_5 = c$$
$$4k_1 + k_2 + 2k_3 + 2k_4 - k_5 = d \Rightarrow$$

$$\begin{bmatrix} 4 & 2 & 3 & 5 & 0 & : & a \\ 4 & 0 & 2 & 6 & 4 & : & b \\ -2 & 0 & -1 & -3 & -2 & : & c \\ 4 & 1 & 2 & 2 & -1 & : & d \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} & 1 & : & \frac{c}{2} \\ 0 & 0 & 0 & 0 & 0 & : & b-2c \\ 0 & 2 & 0 & -4 & -2 & : & b-3c \\ 0 & 1 & 0 & -4 & -6 & : & d-2c \end{bmatrix} \Rightarrow b-2c=0 \Rightarrow$$

$b = 2c$

So the vector (1,1,1,1) is not spanned by the vectors (6,4,-2,4),(2,0,0,1),(3,2,-1,2),(5,6,-3,2),(0,4,-2,-1)

$\therefore$

These vectors do not spans R^4.

(d)**To show that** (1,1,0,0),(1,2,-1,1),(0,0,1,1,),(2,1,2,1) **spans R^4,we let X=(a,b,c,d) be any vector in R^4 .We now seek constants k_1 , k_2,k_3 and k_4 and such that:**

$K_1(1,1,0,0)+k_2(1,2,-1,1)+k_3(0,0,1,1,)+k_4(2,1,2,1)=(a,b,c,d)$

$K_1+k_2+2k_4=a$

$K_1+2k_2+k_4=b$

$-k_2+k_3+2k_4=c$

$K_2+k_3+k_4=d$

$$\begin{bmatrix} 1 & 1 & 0 & 2 & : & a \\ 1 & 2 & 0 & 1 & : & b \\ 0 & -1 & 1 & 2 & : & c \\ 0 & 1 & 1 & 1 & : & d \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 & : & -4a+5b+3c-3d \\ 0 & 1 & 0 & 0 & : & a-b-c+d \\ 0 & 0 & 1 & 0 & : & -3a+3b+2c-d \\ 0 & 0 & 0 & 1 & : & 2a-2b-c+d \end{bmatrix} \Rightarrow$$

$$k_1 = -4a+5b+3c-3d$$
$$k_2 = a-b-c+d$$
$$k_3 = -3a+3b+2c-d$$
$$k_4 = -2a-2b-c+d$$

So the vectors (1,1,0,0),(1,2,-1,1),(0,0,1,1,),(2,1,2,1) **spans R^4** .

4.**To show that** $t^3+2t+1,t^2-t+2,t^3+2,-t^3+t^2-5t+2$ spans P3**,we let X=at^3+bt^2+ct+d be any vector in P_3 .We now seek constants k_1 , k_2,k_3 and k_4 such that:**

$K_1(t^3+2t+1)+k_2(t^2-t+2,t^3+2)+k_3(t^3+2)+ k_4(-t^3+t^2-5t+2)=$**$at^3+bt^2+ct+d$**

$K_1(1,0,2,1)+k_2(0,1,-1,2)+ k_3(1,0,0,2))+ k_4(-1,1,-5,2)=$**(a,b,c,d)**

$$k_1 = -4a+5b+3c-3d$$
$$k_2 = a-b-c+d$$
$$k_3 = -3a+3b+2c-d$$
$$k_4 = -2a-2b-c+d$$

$$\therefore \quad d = \frac{4a+3b-c}{2}$$

So a vector like t^3+t^2+t+1 is not spanned by the vectors$t^3+2t+1,t^2-t+2,t^3+2,-t^3+t^2-5t+2$.

$\therefore$ The vectors $t^3+2t+1,t^2-t+2,t^3+2,-t^3+t^2-5t+2$ do not spans P_3.

5. (a)**To show that**$X_1=(4,2,1),X_2=(2,6,-5),X_3=(1,-2,3)$ are linearly depenent**, we seek constants k_1 , k_2 and k_3 such that:**

$K_1(4,2,1)+k_2(2,6,-5),k_3(1,-2,3)=(0,0,0)$

$4k_1+2k_2+k_3=0$

$2k_1+6k_2-2k_3=0$

$K_1-5k_2+3k_3=0$ $\Rightarrow$

$$\begin{bmatrix} 4 & 2 & 1 & : & 0 \\ 2 & 6 & -2 & : & 0 \\ 1 & -5 & 3 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & \frac{1}{2} & : & 0 \\ 0 & 1 & \frac{-1}{2} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \Rightarrow k_3 = 2t : t \in R, k_2 = \frac{1}{2}k_3 = t, k_1 = \frac{-1}{2}k_2 = -t$$

Take $k_1 = -1, k_2 = 1, k_3 = 2 \Rightarrow$ *The given vectors are linearly dependent*

(b)**To show that**$X_1 =(1,2,-1),X_2=(3,2, 5)$are linearly independent**, we seek constants k_1 and k_2 such that:**

$K_1(1,2,-1)+k_2(3,2,5) = (0,0,0)$ $\Rightarrow$

$$\begin{bmatrix} 1 & 3 & : & 0 \\ 2 & 2 & : & 0 \\ -1 & 5 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \Rightarrow k_1 = 0, k_2 = 0 \Rightarrow$$

The vectors $X_1=(1,2,-1),X_2=(3,2, 5)$ are linearly independent.

(c) **To show that** $X_1 = (1,1,0), X_2 = (0,2,3), X_3 = (1, 2,3), X_4 = (3,6,6)$ are linearly dependent**, we seek constants k_1 , k_2,k_3 and $k_4$such that:**

$K_1(1,1,0)+k_2 (0,2,3)+k_3 (1, 2,3)+k_4 (3,6,6) = (0,0,0) \Rightarrow$

$K_1+k_3+3k_4 = 0$

$K_1+2k_2+ 2k_3+6k_4 = 0$

$3k_2+3k_3+6k_4 =0 \quad \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 3 & :0 \\ 1 & 2 & 2 & 6 & :0 \\ 0 & 3 & 3 & 6 & :0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 2 & : & 0 \\ 0 & 1 & 0 & 1 & : & 0 \\ 0 & 0 & 1 & 1 & : & 0 \end{bmatrix} \Rightarrow$$

Let $k_4=1 \quad \Rightarrow$

$K_2=1$, $k_3=-1$, $k_1 = -2 \quad \Longrightarrow$

The vectors$X_1 = (1,1,0), X_2 = (0,2,3), X_3 = (1, 2,3), X_4 = (3,6,6)$ are linearly dependent.

(d) **To show that**$X_1 = (1,2,3), X_2 = (1,1,1), X_3 = (1,0,1)$ are linearly independent**, we seek constants k_1 , k_2 and $k_3$such that:**

$K_1(1,2,3)+k_2 (1,1,1) +k_3(1,0,1) = (0,0,0) \quad \Longrightarrow$

$K_1+2k_2+k_3 =0$

$2k_1+k_2 = 0$

$3k_1+k_2+k_3 = 0 \quad \Longrightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 2 & 1 & 0 & : & 0 \\ 3 & 1 & 1 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} \Longrightarrow$$

$K_1 = k_2 = k_3 = 0 \quad \Longrightarrow$

The vectors $X_1 = (1,2,3), X_2 = (1,1,1), X_3 = (1,0,1)$ are linearly independent.

3.6.Exercises:

1.

(a) To show that (1,3),(1,-1) is linearly independent ,we form the Equation:

$c_1X_1 + c_2.X_2 = O$

$c_1 (1,3) + c_2.(1,-1) = O$

$(c_1+c_2, 3c_1-c_2) = (0,0),$

$$c_1 + c_2 = 0$$

$$+$$

$$3c_1 - c_2 = 0$$

$$4c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0$$

So (1,3),(1,-1) **is linearly independent.**

To show that (1,3),(1,-1) **spans R^2,we let X=(a,b) be any vector in R^2 .We now seek constants k_1 and k_2 such that**

$K_1(1,3)+k_2.(1,-1)=X=(a,b)$, $(k_1+k_2,3k_1-k_2)=(a,b)$

$$k_1 + k_2 = a$$

$$+$$

$$3k_1 - k_2 = b$$

$$4k_1 = a + b \Rightarrow k_1 = \frac{a+b}{4} \Rightarrow k_2 = a - k_1 = a - \frac{a+b}{4} = \frac{3a+b}{4}$$

So $S=\{(1,3),(1,-1)\}$ spans R^2 and is a basis for R^2.

(b) To show that (1, 2), (2,-3),(3,2) is linearly independent ,weform the equation:

$c_1X_1+c_2.X_2+c_3.X_3=$ O, $c_1(1,2)+c_2.(2,-3)+c_3.(3,2)=(0,0)$

$(c_1+2c_2+3c_3,2c_1-3c_2+2c_3)=(0,0)$,

$c_1+2c_2+3c_3=0$

$2c_1-3c_2+2c_3$

This system has infinity many solutions as

$$c_1=\frac{-13}{7},\quad c_2=\frac{-4}{7},\quad c_3=1$$

Showing that $S=\{(1,2),\ (2,\text{-}3),\ (3,2)\ \}$ **is linearly dependent.**

Hence S does not span R^2 and is not a basis for R^2.

(c) is a basis for R^2.

2. (b)

3.

$$\begin{bmatrix}1 & 1 & -2 & 1\\ 0 & 1 & 0 & 1\\ 1 & 0 & -2 & 0\\ 2 & 3 & -4 & 3\end{bmatrix}\cong\begin{bmatrix}1 & 1 & -2 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Possible answer: $\{t^3+t^2-2t+1,t^2+1\}$ **, dimW=2.4.**

Possible answer: $\{(1,0,1,0),(0,1,-1,0),(0,0,1,1),(0,0,0,1)\}$

5. $$\begin{bmatrix}1 & 0 & 2 & : & 0\\ 2 & 1 & 3 & : & 0\\ 3 & 1 & 2 & : & 0\end{bmatrix}\cong\begin{bmatrix}1 & 0 & 0 & : & 0\\ 0 & 1 & 0 & : & 0\\ 0 & 0 & 1 & : & 0\end{bmatrix}\Rightarrow x_1=x_2=x_3=0$$

The solution set is $S = \{(0,0,0)\}$

The dimension is zero.

3.7.Exercises:

1. $$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

Possible answer : $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2. $$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.308 & 0.615 \\ 0 & 0 & 1 & 0.462 & -0.077 \end{bmatrix} \Rightarrow$$

The row and column rank are 3

3. $$S = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 5 & -5 \\ 2 & -1 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0.833 \\ 0 & 1 & -1.333 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

Linearly dependent.

4. (a) $$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} . \cong \begin{bmatrix} 1 & 0 & 0 & 0.265 \\ 0 & 1 & 0 & 0.675 \\ 0 & 0 & 1 & -0.723 \end{bmatrix} \Rightarrow$$

The rank of the coefficient and augmented matrices are 3 ,

So the linear system has a solution.

(b) $$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} . \cong \begin{bmatrix} 1 & 0 & -1 & 1.143 \\ 0 & 1 & 1 & -1.429 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the coefficient matrix is 2.

$$\begin{bmatrix} 1 & -2 & -3 & 4 & :1 \\ 4 & -1 & -5 & 6 & :2 \\ 2 & 3 & 1 & -2 & :2 \end{bmatrix} . \cong \begin{bmatrix} 1 & 0 & -1 & 1.143 & : & 0.571 \\ 0 & 1 & 1 & -1.429 & : & 0.286 \\ 0 & 0 & 0 & 0 & : & 1 \end{bmatrix}$$

The rank of the augmented matrix coefficient matrix is 3.

So the linear system has no solution.

3.9Glossary:

Vector:متجه

N-dimensional:n ذو بعد

Linear Space:فضاء خطي

Subspace: فضاء جزئي

Spanning Set: مجموعة مولدة

Linearly Independent:مستقلة خطياً

Linearly dependent:مرتبطة خطياً

Polynomial:كثير حدود

Linear Combination:تركيبة خطية

Basis:أساس

Dimension:بعد

Chapter 4

Linear Transformations and Matrices

4.1. Exercises:

1. Which of the following are linear transformation?

(a) L $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$

(b) L $\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2+x \\ y-y^2 \end{bmatrix}$

2. Which of the following are linear transformation?

(a) L $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 0 \\ 2x-z \end{bmatrix}$

(b) L $\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2-y^2 \\ x^2+y^2 \end{bmatrix}$

3. Let: $L: R^2 \to R^2$ be linear transformation for which we know that:

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad and \quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(a) What is $L\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$?

(b) What is $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$?

4. Let L: $p_2 \to p_3$ be linear transformation for which we know that L(1)=1 , L(t) =t^2 , and L(t^2) =t^3+t .

(a) . Find L($2t^2$-5t+3).

(b) .Find L(at^2+bt+c).

4.2. Exercises:

1. Let L: $R^2 \to R^3$ be defined by?

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix}$$

(a). Find ker L.

(b) . Is L one –to –one?

(c) . Is L onto?

2. Let L : $R^5 \to R^4$ be defined by ?

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

(a) Find a basis for kerL .
(b) Find a basis for range L .
(c) Verify Theorem 4.7.

3. Let L : $R^4 \to R^6$ be Linear transformation ?

(a) If dim (ker L) =2 ,what is dim (range L)?

(b)) If dim (range L) =3 ,what is dim (ker L)?

4.Determine the dimension of the solution space to the homogeneous system AX=0 for the given matrix.

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 4 & -1 \end{bmatrix}$$

A =

4.3. Exercises:

1. Let S = $\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$

be abasis for R^2 . Find the coordinate vectors of the following vectors with respect to S .

(a) . $\begin{bmatrix}-3\\-7\end{bmatrix}$

(b) . $\begin{bmatrix}3\\7\end{bmatrix}$

2. Let $S=\{t^2+1,t-1,t\}$

be a basis for P_2. Find the coordinate vectors of the following vectors with respect to S .

(a) $t^2 + 2$

(b) 2t – 1

3. Let L : $R^2 \rightarrow R^3$ be defined by

$$L\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x-2y\\2x+y\\x+y\end{bmatrix}$$

Let S and T be the natural bases for R^2 and R^3 , respectively . Also , Let

$$S'=\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}\quad and \quad T'=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

be bases for R^2 and R^3 , respectively, Find the matrix of L with respect to:

(a) S and T.

(b) $S' \quad and \quad T'$

(c)Compute $L\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ $using\ the\ matrices\ obtained\ \ in\ (a)\ \ and\ (b).$

4. Let L : $R^3 \to R^2$ be defined by

$$L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+y\\y-z\end{bmatrix}$$

Let S and T be the natural bases for R^3 and R^2, respectively. Also, let

$$S' = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\} \quad and \quad T' = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$$

be bases for $R^3$and R^2, respectively.

Find the representation of L with respect to

(a) S and T

(b) $S' \quad and \quad T'$

(c) Compute $L\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$ using both representations.

5.Let $L: R^2 \to R^2$ be a linear transformation. Suppose that thematrix of L with respect to the basis

$$S = \{X_1, X_2\}$$

is

$$A = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix}$$

where

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad and \qquad X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a)Compute $[L(X_1)]_S \quad and \quad [L(X_2)]_S$

(b) Compute $L(X_1) \quad and \quad L(X_2)$

(c) Compute $L\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$

6. LetL : $R^3 \rightarrow R^3$ be defined by

$$L = \left(\begin{bmatrix} 1 \backslash \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a). Find the matrix of L with respect to the natural basis S for R^3 .

(b). *Find* $L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ Using the definition of L and also using the matrix obtained in (a).

4.4 Answers to Exercises:

4.1.Exercises:

1. (a)

Let

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Then

$$L(X+Y) = L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = L\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ y_1 + y_2 \\ x_1 + x_2 - z_1 - z_2 \end{bmatrix}$$

$$L(X+Y) \quad = \begin{bmatrix} x_1 + y_1 \\ y_1 \\ x_1 - z_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ y_2 \\ x_2 - z_2 \end{bmatrix} = L(X) + L(Y)$$

Also, if c is a real number, then

$$L(cX) = L\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + cy_1 \\ cy_1 \\ cx_1 - cz_1 \end{bmatrix} = c.\begin{bmatrix} x_1 + y_1 \\ y_1 \\ x_1 - z_1 \end{bmatrix} = c.L(X)$$

Hence L is a linear transformation.

(b) Let

$$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$L(X+Y) = L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = L\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2)^2 + x_1 + x_2 \\ y_1 + y_2 - (y_1 + y_2)^2 \end{bmatrix}$$

$$L(X) + L(Y) = \begin{bmatrix} x^2{}_1 + x_1 \\ y_1 - y^2{}_1 \end{bmatrix} + \begin{bmatrix} x^2{}_2 + x_2 \\ y_2 - y^2{}_2 \end{bmatrix} = \begin{bmatrix} x^2{}_1 + x_1 + x^2{}_2 + x_2 \\ y_1 - y^2{}_1 + y_2 - y^2{}_2 \end{bmatrix} \neq L(X+Y)$$

Hence L is not a linear transformation.

2. Let

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$L(X+Y) = L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = L\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ 0 \\ 2(x_1 + x_2) - (z_1 + z_2) \end{bmatrix}$$

$$L(X+Y) = \begin{bmatrix} x_1 + y_1 \\ 0 \\ 2x_1 - z_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ 0 \\ 2x_2 - z_2 \end{bmatrix} = L(X) + L(Y)$$

Also, if c is a real number, then

$$L(cX) = L\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + cy_1 \\ 0 \\ 2cx_1 - cz_1 \end{bmatrix} = c.\begin{bmatrix} x_1 + y_1 \\ 0 \\ 2x_1 - z_1 \end{bmatrix} = c.L(X)$$

Hence L is a linear transformation.

(b) $L(X) + L(Y) \neq L(X) + L(Y)$

Hence L is not a linear transformation.

3. (a) $(3,-2) = c_1(1,1) + c_2(0,1) = (c_1, c_1 + c_2) \Rightarrow c_1 = 3, c_2 = -5$

$$L\begin{bmatrix} 3 \\ -2 \end{bmatrix} = c_1.L\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2.L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3.L\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3.\begin{bmatrix} 2 \\ -3 \end{bmatrix} - 5.\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -19 \end{bmatrix}$$

(b) $$(a,b) = c_1(1,1) + c_2(0,1) = (c_1, c_1 + c_2) \Rightarrow c_1 = a, c_2 = b - a$$

$$L\begin{bmatrix} a \\ b \end{bmatrix} = c_1.L\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2.L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a.L\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L\begin{bmatrix} a \\ b \end{bmatrix} = a.\begin{bmatrix} 2 \\ -3 \end{bmatrix} + (b-a).\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a+b \\ -5a+2b \end{bmatrix}$$

We can solve this question using $L[X] = AX \Rightarrow$

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_2 \\ c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow c_1 + c_2 = 2 \quad , \quad c_3 + c_4 = -3$$

$$L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_2 = 1 \quad , \quad c_4 = 2 \quad ,$$

$$c_1 = 2 - c_2 = 1, c_3 = -3 - c_4 = -5 \Rightarrow$$

$$A = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}.\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -19 \end{bmatrix}$$

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ -5a+2b \end{bmatrix}$$

4. (a) $L(2t^2-5t+3)=2.L(t^2)-5L(t)+3.L(1)$

$L(2t^2-5t+3)= 2.(t^3+t)-5(t^2)+3.1 = 2t^3-5t^2+2t+3$

(b) $L(at^2+bt+c)=a.L(t^2)+bL(t)+c.L(1)$

$L(at^2+bt+c)= a.(t^3+t)+b(t^2)+c.1 = at^3+bt^2+at+c$

4.2.Exercises:

1. (a) x = 0

x+y = 0

y = 0

x=y=0, $\ker L = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

(b) L is one - to – one because $\ker L = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

(c) Given any $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ *in* R^3, *where* y_1, y_2 *and* y_3

are any real numbers, can we find $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

in R^2 so that $L([x]) = Y$? we are seeking a solution to the linear system:

$$L(X) = A.X = Y$$

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \\ c_5 & c_6 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix} = \begin{bmatrix} c_1x + c_2y \\ c_3x + c_4y \\ c_5x + c_6y \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix} \Rightarrow$$

$$c_1x + c_2y = x \Rightarrow c_1 = 1, c_2 = 0$$
$$c_3x + c_4y = x + y \Rightarrow c_3 = 1, c_4 = 1$$
$$c_5x + c_6y = y \Rightarrow c_5 = 0, c_6 = 1 \Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We are seeking a solution to the linear system :

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The reduced row echelon form of the augmented matrix is :

$$\begin{bmatrix} 1 & 0 & : & y_1 \\ 0 & 1 & : & y_2 - y_1 \\ 0 & 0 & : & y_3 - y_2 + y_1 \end{bmatrix}$$

Thus a solution exists only for $y_3 - y_2 + y_1 = 0,$

and so L is not onto.

2.(a):

$$\begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 2 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -0.5 \end{bmatrix} \Rightarrow$$

$$x_5 = 0 \quad , \quad x_4 = t : t \in R, \quad x_3 = x_4 = t : t \in R$$

$$x_2 = s : s \in R, \quad x_1 = -2x_4 = -2t : t \in R$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} = t.\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s.\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$posssible \quad basis \quad for \quad \ker L : \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(b)

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 3 & 2 & 5 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$posssible \quad basis \quad for \quad rangeL : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(c) L:

dim(ker L)+dim(range L)=dim V

2 + 3 = 5

3. (a) dim(range L)=4-2=2

(b) dim(ker L)=4-3=1

4. $$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 4 & -1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$x_3 = s : s \in R, \quad x_1 = 7x_3 = 7s : s \in R$$
$$x_2 = -5x_3 = -5s : s \in R$$

The solution space: $\left\{ \begin{bmatrix} 7s \\ -5s \\ s \end{bmatrix} : s \in R \right\}$

The dimension of the solution space to the homogeneous system AX=0 for the given matrix A is 1.

4.3. Exercises:

1. (a) $$\begin{bmatrix} -3 \\ -7 \end{bmatrix} = c_1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2.\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ -c_1 + 3c_2 \end{bmatrix} \Rightarrow$$

$$c_1 + 2c_2 = -3$$
$$+$$
$$-c_1 + 3c_2 = -7$$

$$5c_2 = -10 \Rightarrow c_2 = -2 \Rightarrow c_1 = 7 + 3c_2 = 7 - 6 = 1$$

The coordinate vector of the vector $\begin{bmatrix} -3 \\ -7 \end{bmatrix}$ with respect to S : $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = c_1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2.\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ -c_1 + 3c_2 \end{bmatrix} \Rightarrow$$

$$c_1 + 2c_2 = 3$$

$$+$$
$$-c_1 + 3c_2 = 7$$

The coordinate vector of the vector $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ with respect to S : $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

2. (a) $t^2 + 2 = c_1(t^2+1)+c_2(t-1)+c_3(t)=c_1.t^2+(c_2+c_3)t+(c_1-c_2)$

$c_1 = 1$, $c_1-c_2 = 2$, $c_2 = -1$, $c_2+c_3 = 0$, $c_3 = -c_2 = 1$

The coordinate vector of the vector $t^2 + 2$ with respect

To S : $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. (a)

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

The matrix of L with respect to S and T:

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) $$L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = c_1.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3.\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{7}{3}$$

$$c_2 = -\frac{2}{3}, \qquad c_3 = \frac{2}{3}$$

$$L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\\1\end{bmatrix} = c_1.\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2.\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3.\begin{bmatrix}1\\-1\\1\end{bmatrix} \Rightarrow c_1 = \frac{-4}{3}$$

$$c_2 = \frac{5}{3}, \quad c_3 = -\frac{2}{3}$$

The matrix of L with respect to S` and T` : $\begin{bmatrix}\frac{7}{3} & \frac{-4}{3}\\ \frac{-2}{3} & \frac{5}{3}\\ \frac{2}{3} & \frac{-2}{3}\end{bmatrix}$

(c)- For (a) :

$$L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1 & -2\\2 & 1\\1 & 1\end{bmatrix}.\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-3\\4\\3\end{bmatrix}$$

- For(b):

$$L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}\frac{7}{3} & \frac{-4}{3}\\ \frac{-2}{3} & \frac{5}{3}\\ \frac{2}{3} & \frac{-2}{3}\end{bmatrix}.\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\frac{-1}{3}\\ \frac{8}{3}\\ \frac{-2}{3}\end{bmatrix}$$

4. (a) $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix},$

$$L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix},$$

The representation of L with respect to S and T:

$$[L(X)]_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} . [X]_S$$

(b) $$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1 . \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} \Rightarrow$$

$$-c_1 + c_2 = 2 \quad , \quad 2c_2 + c_1 = 1 \Rightarrow c_1 = -1, \quad c_2 = 1$$

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)_{T'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 . \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} \Rightarrow$$

$$-c_1 + c_2 = 1 \quad , \quad 2c_2 + c_1 = 1 \Rightarrow c_1 = \frac{-1}{3}, \quad c_2 = \frac{2}{3}$$

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)_{T'} = \begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1 . \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} \Rightarrow c_1 = 0, c_2 = 0$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)_{T'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The representation of L with respect to: $S' \quad and \quad T'$

$$[L(X)]_{T'} = \begin{bmatrix} -1 & \frac{-1}{3} & 0 \\ 1 & \frac{2}{3} & 0 \end{bmatrix} . [X]_{S'}$$

(c) - For (a) :

$$L\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

- For (b) :

$$L\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{3} & 0 \\ 1 & \frac{2}{3} & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{-5}{3} \\ \frac{7}{3} \end{bmatrix}$$

5.(a) $[L(X_1)]_s = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$[L(X_2)]_s = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

(b) $L(X_1) = 2.X_1 - 1.X_2 = 2.\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$L(X_2) = -3.X_1 + 4.X_2 = -3.\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 3 \end{bmatrix} = c_1 X_1 + c_2 . X_2$

$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = c_1 . \begin{bmatrix} 1 \\ 2 \end{bmatrix} . c_2 . \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - c_2 \end{bmatrix} \Rightarrow$

$$c_1 + c_2 = -2 \quad , \quad 2c_1 - c_2 = 3 \Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{-7}{3}$$

$$L\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = c_1.L([X_1]) + c_2.L([X_2])$$

(6)(a)

$$L\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \frac{1}{3}.\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{-7}{3}.\begin{bmatrix} 1 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ 25 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) -Using the definition of L:

$$L\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} = 1 * L\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} + 2 * L\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} + 3 * L\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$L\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} = 1 * \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 * \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 3 * \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$$

- **Using the matrix obtained in (a):**

$$L(X) = A * X$$

$$L\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$$

4.5 Glossary:

Linear Transformation: تحويل خطي

Kernel: النواة

Range:المدى

One to One:واحد لواحد

Onto:شامل

Reflection: انعكاس

Rotation:دوران

Composition of Linear Transformation:تركيب التحويلات الخطية

Transition Matrix: المصفوفة الناقلة

Matrix Representation of a Linear Transformation:تمثيل التحويل الخطي بمصفوفة

Chapter 5

Eigenvalues and Eigenvector's

5.1.Exercises:

1. Find the characteristic polynomial of each matrix .

(a) . $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$

(b). $\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

2. Find the characteristic polynomial,eigenvalues, and eigenvectors of each matrix**.**

(a) . $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

3. Find which of the matrices are diagonalizable.

(a) . $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

4. Find for each matrix A , if possible , a non-singular matrix Psuch that $P^{-1}AP$ is diagonal .

(a). $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

5. Prove that if A and B are similar matrices, they have the same characteristic polynomials and hence the same eigenvalues.

5.2.Exercises:

1. Verify that

$$P = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Is an orthogonal matrix.

2.Diagonalizable each given matrix A and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

(a) . $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(b). $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

3.*Diagonalizable each given matrix***.**

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(a) .

(b). $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) . $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

4 . Show that if A is an orthogonal matrix, then $|A| = \pm 1.$

5*.Show that if A is an orthogonal matrix , then* A^{-1}.

is orthogonal.

5.3 Answers to Exercises:

5.1. Exercises:

1. (a)

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -1 \\ 0 & \lambda - 1 & -2 \\ 1 & -3 & \lambda - 2 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 7$$

(a) $$|\lambda I - A| = \begin{vmatrix} \lambda - 4 & 1 & -1 \\ 0 & \lambda - 2 & -1 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 4)(\lambda - 2)(\lambda - 3) = \lambda^3 - 9\lambda^2 + 26\lambda - 24$$

2. (a) $$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 1 & \lambda - 3 & 0 \\ -3 & -2 & \lambda + 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda + 2) = f(\lambda)$$

$$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -2$$

$$(\lambda_1 I - A)X_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & : & 0 \\ 1 & -2 & 0 & : & 0 \\ -3 & -2 & 3 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & \frac{-3}{4} & : & 0 \\ 0 & 1 & \frac{-3}{8} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \Rightarrow$$

$$x_3 = s : s \in R, \quad x_1 = \frac{3}{4} x_3 = \frac{3}{4} s, \Rightarrow x_2 = \frac{3}{8} x_3 = \frac{3}{8} s$$

$$Let \quad s = 8 \Rightarrow x_1 = 6, \quad x_2 = 3, \Rightarrow x_3 = 8 \Rightarrow \quad X_1 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$$

$$(\lambda_2 I - A)X_2 = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 0 & : & 0 \\ 1 & 0 & 0 & : & 0 \\ -3 & -2 & 5 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & \frac{-5}{2} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \Rightarrow$$

$$x_3 = s : s \in R, \quad x_1 = 0, \Rightarrow x_2 = \frac{5}{2} x_3 = \frac{5}{2} s$$

$$(\lambda_3 I - A)X_3 = 0 \Rightarrow \begin{bmatrix} -3 & 0 & 0 & : & 0 \\ 1 & -5 & 0 & : & 0 \\ -3 & -2 & 0 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \Rightarrow$$

$x_3 = s : s \in R, \quad x_1 = 0, \Rightarrow x_2 = 0$

$Let \quad s = 1 \Rightarrow x_1 = 0, \quad x_2 = 0, \Rightarrow x_3 = 1 \Rightarrow \qquad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2 = \lambda^2 - 5\lambda + 6 = f(\lambda)$

$\lambda_1 = 2, \lambda_2 = 3,$

$$(\lambda_1 I - A)X_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & : & 0 \\ -2 & -2 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \Rightarrow$$

$x_2 = s : s \in R, \quad x_1 = -x_2 \Rightarrow$

$Let \quad s = 1 \Rightarrow x_1 = -1, \quad x_2 = 1, \Rightarrow \qquad X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$(\lambda_2 I - A)X_2 = 0 \Rightarrow \begin{bmatrix} 2 & 1 & : & 0 \\ -2 & -1 & : & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & \frac{1}{2} & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \Rightarrow$$

$x_2 = s : s \in R, \quad x_1 = -\frac{1}{2} x_2 \Rightarrow$

$Let \quad s = 2 \Rightarrow x_1 = -1, \quad x_2 = 2, \Rightarrow \qquad X_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

3. (a) $Eigenvals(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Eigenvecs(A) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow By \quad using \quad theorem 5.2 \quad the$ $matrix \quad is \quad not \quad diagonalizable$

(b) $Eigenvals(A)=\begin{bmatrix}1\\-1\\2\end{bmatrix},\quad Eigenvecs(A)=\begin{bmatrix}1&-0.707&0.964\\0&0.707&0.148\\0&0&0.222\end{bmatrix}\Rightarrow by\quad u\sin g$

$theorem 5.2\quad the\quad matrix\quad is\quad Diagonalizable$

4. (a) $Eigenvals(A)=\begin{bmatrix}3\\1\\1\end{bmatrix},\quad Eigenvecs(A)=\begin{bmatrix}0.811&-0.707\\0.324&0\\-0.487&0.707\end{bmatrix}\Rightarrow by\quad u\sin g\quad theorem 5.2$

$the\quad matrix\quad is\quad not\quad diagonalizable$

(b) $Eigenvals(A)=\begin{bmatrix}-1\\4\\1\end{bmatrix},\quad Eigenvecs(A)=\begin{bmatrix}-0.832&-0.707&-0.137\\0&0&0.824\\0.555&-0.707&-0.549\end{bmatrix}\Rightarrow$

$Diagonalizable\Rightarrow U\sin g\quad theorem 5.2:$

$$P=\begin{bmatrix}-0.832&-0.707&-0.137\\0&0&0.824\\0.555&-0.707&-0.549\end{bmatrix}\quad or\quad P=\begin{bmatrix}-3&-1&-1\\0&0&6\\2&-1&-4\end{bmatrix}$$

5. $B=P^{-1}.A.P\Rightarrow|\lambda I-B|=|\lambda I-P^{-1}A.P|=|P^{-1}\lambda IP-P^{-1}A.P|=|P^{-1}(\lambda.I-A)P|=$

$$=|P^{-1}|.|\lambda I-A|.|P|=\frac{1}{|P|}.|\lambda I-A|.|P|=|\lambda I-A|\Rightarrow$$

$$|\lambda I-B|=|\lambda I-A|$$

So A and B have the same characteristic polynomials and hence the same eigenvalues.

5.2. Exercises:

1.

$$P^T=P^{-1}=\begin{bmatrix}\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{-2}{3}&\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{-2}{3}&\frac{2}{3}\end{bmatrix}\Rightarrow$$

P Is orthogonalmatrix.

2. (a)

$$eigenvals(A) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad eigenvecs(A) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b)

$$eigenvals(A) = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad eigenvecs(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. (a)

$$|\lambda I - A| = 0 = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} \Rightarrow$$

$$(\lambda - 2)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 3 \Rightarrow$$

$$eigenvals(A) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \quad U \sin g \quad theorem 5.2 \Rightarrow$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$|\lambda I - A| = 0 = \begin{vmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} \Rightarrow$$

$$(\lambda-1).\begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-1)^2 - 1\right] = 0 \Rightarrow (\lambda-1)(\lambda^2 - 2\lambda) = 0$$

$$(\lambda-1)(\lambda-2)\lambda = 0 \Rightarrow$$

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 0$$

$$eigenvals(A) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}. \quad U\sin g \quad theorem 5.2 \Rightarrow$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$$|\lambda I - A| = 0 = \begin{vmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{vmatrix} \Rightarrow$$

$$(\lambda-2).\begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} + 1*\begin{vmatrix} -1 & -1 \\ -1 & \lambda-2 \end{vmatrix} - 1*\begin{vmatrix} -1 & \lambda-2 \\ -1 & -1 \end{vmatrix} = 0$$

$$(\lambda-2)\left[(\lambda-2)^2 - 1\right] - 1*(\lambda-2) - 1 - 1 - \lambda + 2 = 0 \Rightarrow$$

$$(\lambda-2)(\lambda^2 - 4\lambda + 3) - \lambda + 2 - \lambda = 0$$

$(\lambda-1)^2(\lambda-4)=0 \Rightarrow$

$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 4$

$$eigenvals(A) = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}. \quad U\sin g \quad theorem 5.2 \Rightarrow$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4. if A is an orthogonal matrix **,then $A^{-1} = A^t$**

$$So \quad |A^{-1}| = |A^T| \Rightarrow |A^{-1}| = |A|$$

$$\frac{1}{|A|} = |A| \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$$

5.Need to show that $(A^{-1})^{-1} = (A^{-1})^T$

Suppose that*A is an orthogonal matrix* $\Rightarrow A^{-1} = A^T \text{........}(a)$

By taking the transpose both sides of equation(1)we get $(A^{-1})^T = (A^T)^T = A = (A^{-1})^{-1}$, $(A^{-1})^{-1} = (A^{-1})^T$

$$\therefore \quad A \quad is \quad an \quad orthogonal \quad matrix$$

5.4 Glossary:

Eigen Value: قيمة مميزة

Eigen Vector:متجه مميز

Characteristic Polynomial:كثير الحدود المميز

Characteristic Equation:المعادلة المميزة

Eigen Space:الفضاء المميز

Similarity: التشابه

Similar matrices:مصفوفات متشابهة

Diagonalizable Matrix: المصفوفة القابلة للتحويل الى مصفوفة قطرية

Chapter 6

Linear Programming

6.1.Exercises:

1. Steel producer makes two types of steel: regular and special .Aton of regular steel requires 2 hours in the open –hearth furnace and 5 hours in the soaking pit ;a ton of special steel requires 2 hours in the open-hearth furnace and 3 hours in the soaking pit. The open- hearth furnace is available 8 hours per day and the soaking pit is available 15 hours per day. The profit on a tonof regular steel is$120 and it is $100 on a ton of special steel.Determine how many tons of each type of steel should be made to maximize the profit.

2.Atelevision producer designsa program based on a comedian and time for commercials .The advertiser insists on at least 2 minutes of advertising time , the station insists on no more than 4 minutes of advertising time, and the comedian insists on at least 24 minutes of the comedy program .Also , the total time allotted for the advertising and comedy portions of the program cannot exceed 30 minutes .If it has been determined that each minute of advertising(very creative) attracts 40,000 viewers and each minute of the comedy program attracts 45,000 viewers, how should the time be divided between advertising and a programming to maximize the number of viewer-minutes ?

3. The protein Diet Club serves a luncheon consisting of two dishes, A and B . Suppose that each unit of A has 1 gram of fat , 1 gram of carbohydrate , and 4 grams of protein , whereas each unit of B has 2 grams of fat, 1 gram of carbohydrate, and 6 grams of protein .If the dietician planning the luncheon wants to provide no more than 10 grams of fat or more than 7 grams of carbohydrate, how many units of A and how many units of B should be served to maximize the amount of protein consumed ?

4. Sketch the set of points satisfying the given system of inequalities.

a.

$$\begin{aligned} 2x - y &\le 6 \\ 2x + y &\le 10 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

b.

$$\begin{aligned} x + y &\ge 4 \\ x + 4y &\ge 8 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

5. Minimize z = 3x - y

Subject to

$$\begin{aligned} -3x + 2y &\le 6 \\ 5x + 4y &\ge 20 \\ 8x + 3y &\le 24 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

6 .Formulate the following linear programming problem as a new problem with slack variables.

Maximize $z = 2x_1 + 3x_2 + 7x_3$

subject to

$$\begin{aligned} 3x_1 + x_2 - 4x_3 &\le 3 \\ x_1 - 2x_2 + 6x_3 &\le 21 \\ x_1 - x_2 - x_3 &\le 9 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_3 &\ge 0 \end{aligned}$$

6.2 Answers to Exercises:

6.1. Exercises

1.Supposethat x is the mount of regular steel, and y is the mount of special steel.

Maximize Z= 120 x + 100 y

Subject to

$$2x + 2y \leq 8$$
$$5x + 3y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

2.Suppose that x is advertising time, and y is the time of the comedy program

Maximize Z= 40000 x +45000 y

Subject to

$$x + y \leq 30$$
$$y \geq 24$$
$$x \geq 2$$
$$x \leq 4$$
$$x \geq 0$$
$$y \geq 0$$

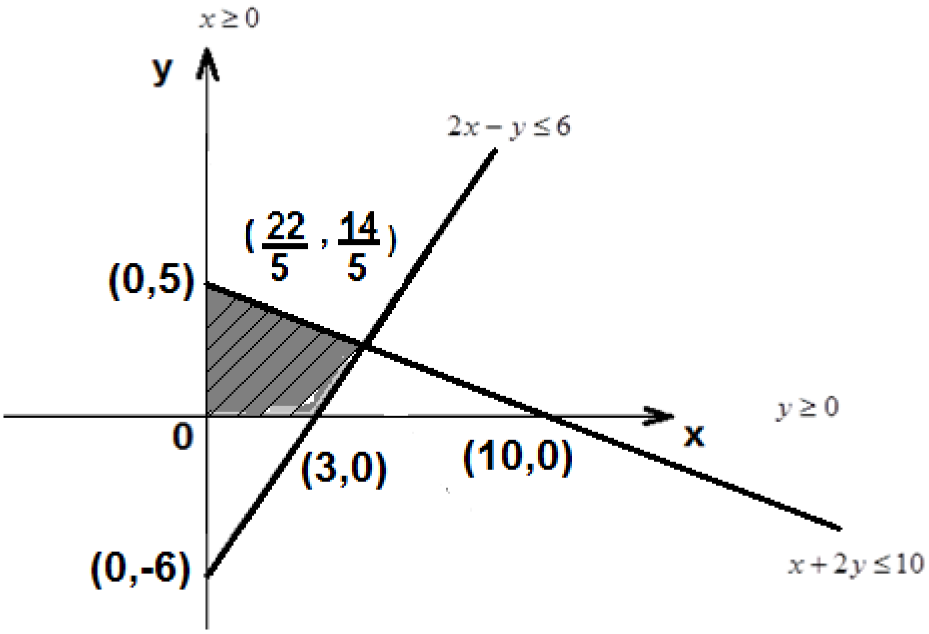
3.Suppose that x is the mount units of type A ,and y is the the mount units of type B.

Maximize Z = 4x+6y

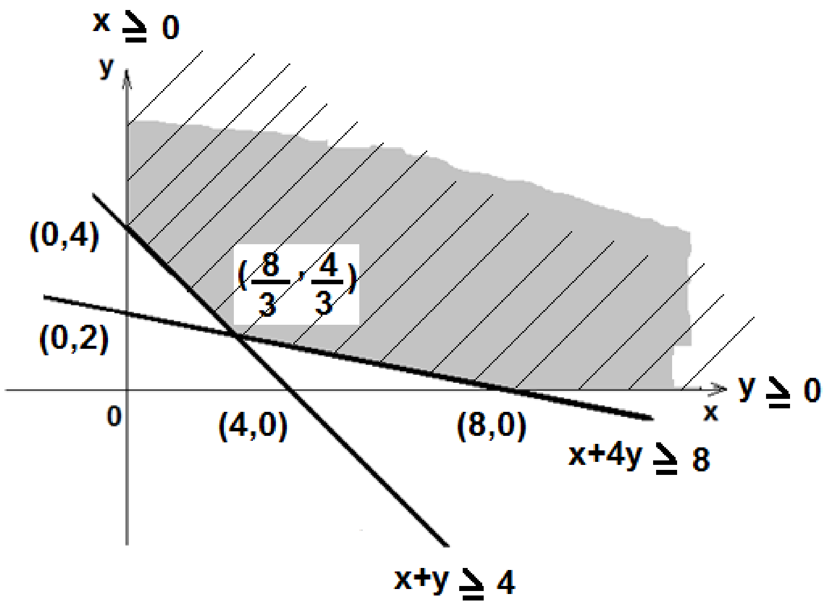
Subject to

$$x + 2y \le 10$$
$$x + y \le 7$$
$$x \ge 0$$
$$y \ge 0$$

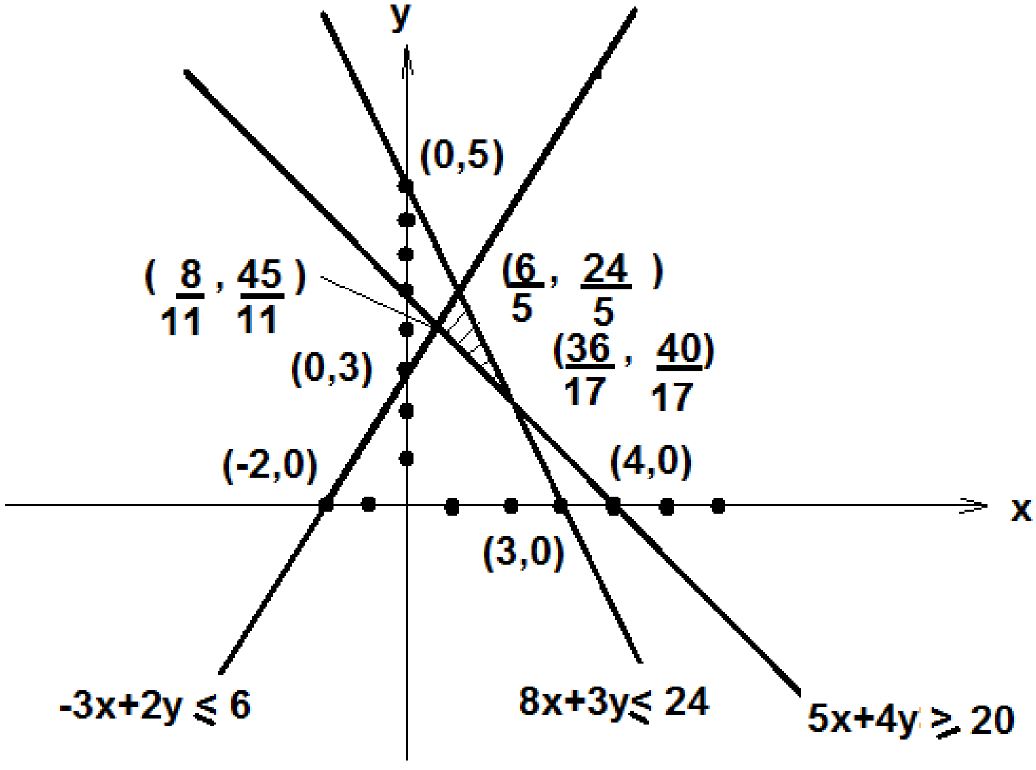
4. (a)



(b)



5.



The point	Z
$(\frac{8}{11},\frac{45}{11})$	$\frac{-21}{11}$

$(\frac{36}{17},\frac{40}{17})$	$(\frac{68}{17})$
$(\frac{6}{5},\frac{24}{5})$	$\frac{-6}{5}$

$x=\frac{8}{11}$, $y=\frac{45}{11}$

$\min z=\frac{-21}{11}$

6.

Maximize z = $2x_1 + 3x_2 + 7x_3$

Subject to

$$3x_1 + x_2 - 4x_3 + x_4 = 3$$
$$x_1 - 2x_2 + 6x_3 + x_5 = 21$$
$$x_1 - x_2 - x_3 + x_6 = 9$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$$

6.3 Glossary:

Linear Inequality:المتباينة الخطية

Objective Function:اقتران الهدف

Constraints:(شروط) قيود

The End